On functions as transformations and the transformation of functions

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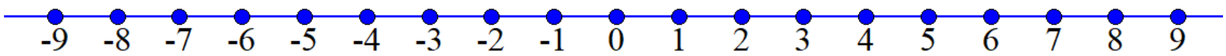
1.1 Functions as transformations

The usual way of thinking about functions is as a formula. We put in a number *x* and it gives us the answer *y*. In this section I will describe another conception of functions, this conception addressing a fundamental property not only of functions but also of other areas of maths such as algebra, diffentiation, integration, complex numbers, matrices, etc... One might even say that this is a unifying conception which unifies all these (and many more) areas of maths.

To start, let us consider the number line. This increases forever towards the right hand side and decreases forever towards the left hand side. What we will look at in these notes will apply to all real numbers (integers, rational and irrational numbers), but to make things easier let us look only at integer numbers:



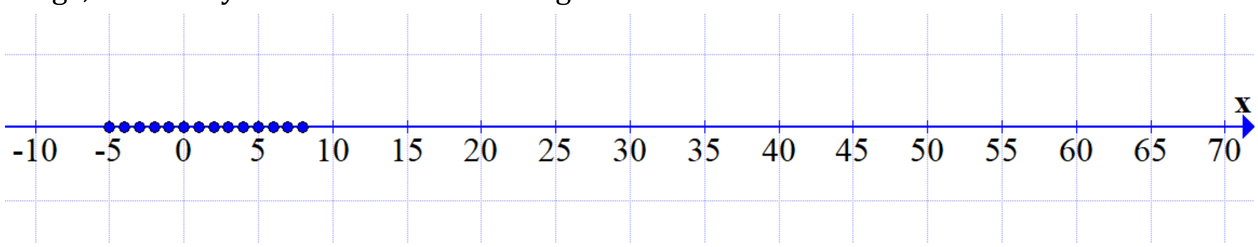
This line shows us how the integers are distributed. In this case the numbers are evenly distributed, and are spaced 1 unit apart from each other. This number line is what is referred to as the *x*-axis on a graph.

One way of conceiving of functions is that

functions transform the distribution of numbers on the number line.

That's it. I could stop my notes here. However, it might be useful to explain this in more detail.

So consider the function $f(x) = x^2$. This is a functions which takes each number from the number line and transforms it into its squared number. Let us take -5 to 8 as a representative range, as seen by the blue dots in the diagram below:



When we square these numbers they get transformed into those shown by the red dots below:

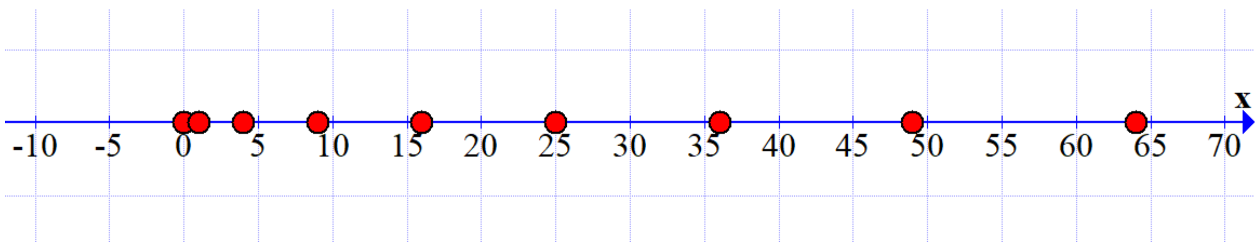
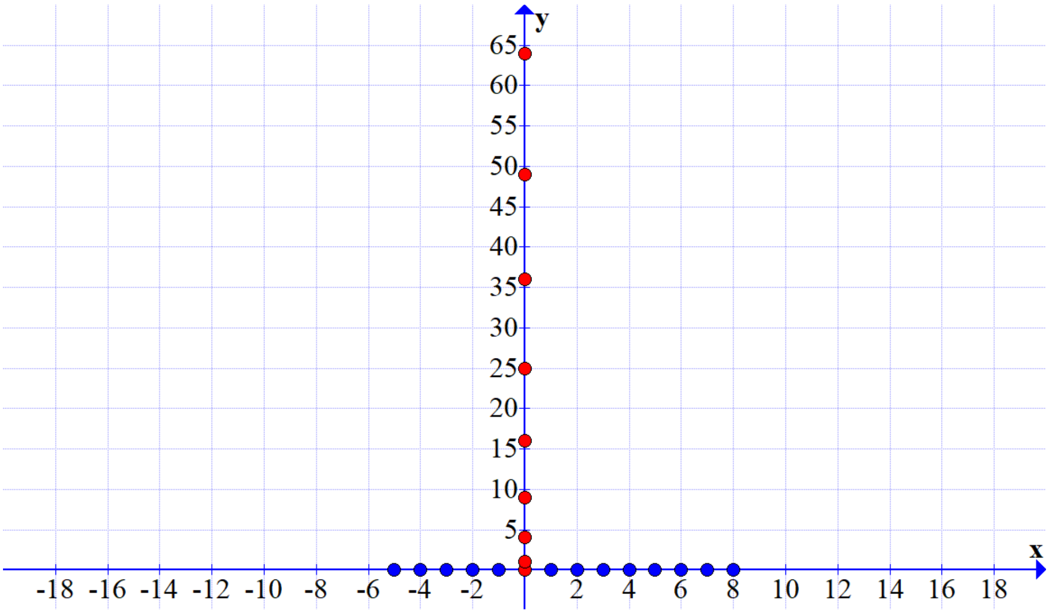


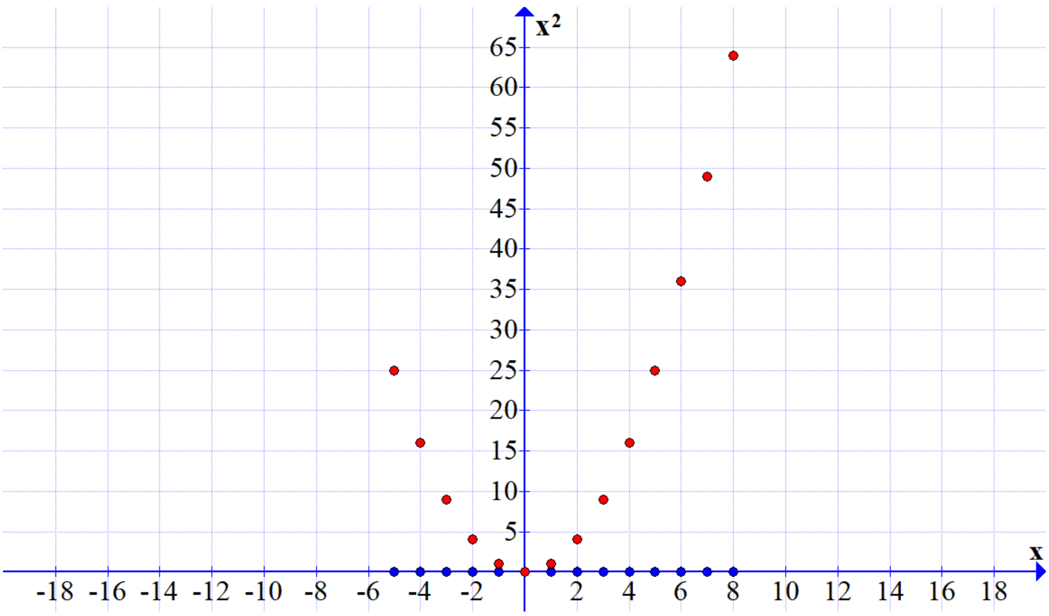
diagram 1

Visually speaking these *x* numbers have been stretched out. This particular pattern stretching represents the squaring effect. So this is what the *x* numbers look like when they get transformed by squaring. We also notice that there are no red dots on the negative side of the *x*-axis. So this is another effect of the "squaring" transformation.

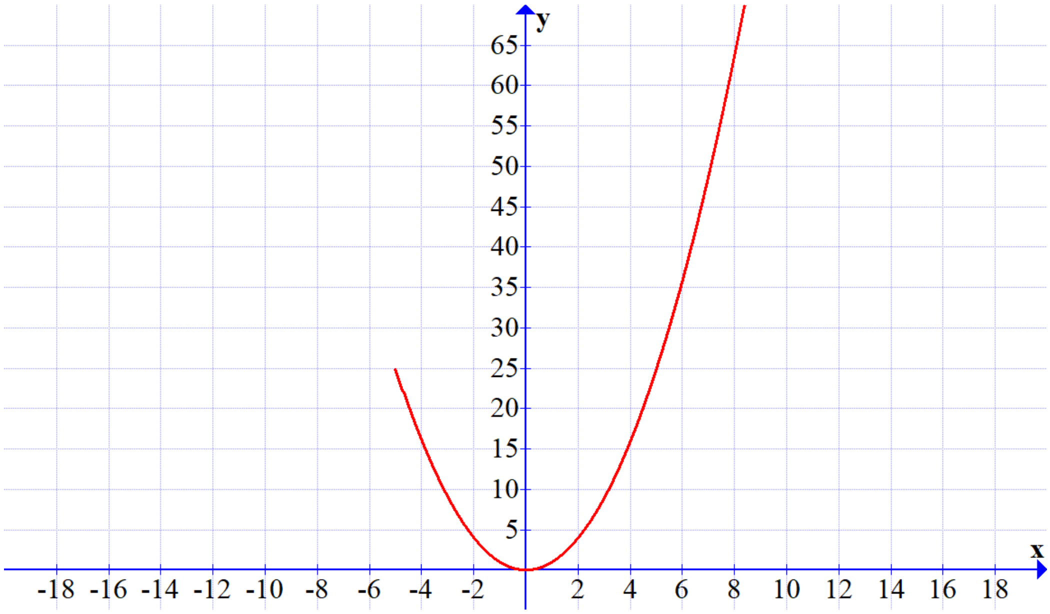
However, plotting the original number line and the transformed number line as two separate diagrams is not the best way to visually represent the distribution of the *x* numbers under an x^2 transformation. So we plot the distribution of the transformed numbers vertically:



But even by this visual representation it is not easy to see the transformation effect. So, instead, we place each transformed *x* number directly over its orignial untransformed *x* number:

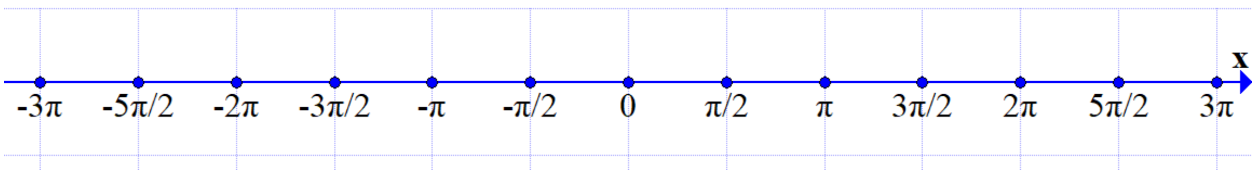


In other words:

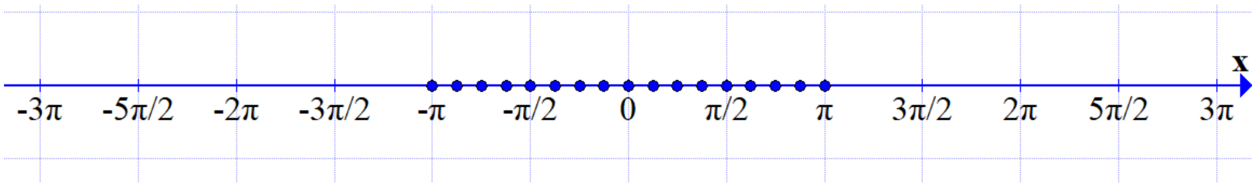


This graph shows the usual curve of $f(x) = x^2$. Notice that this curve is just a simple and easy way of visualising the transformation shown in diagram 1, but that the real underlying effect of squaring our numbers (i.e. of doing $f(x) = x^2$) is to transform the distribution of numbers on the number line into another distribution.

This idea applies to all functions. For example, if we consider the distribution of numbers in terms of fractions of π we get the number line shown below:

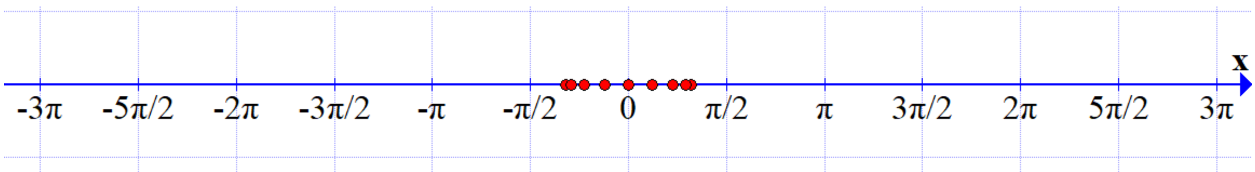


Suppose we now take an interval of $-\pi$ to π, as shown by the blue dots below:

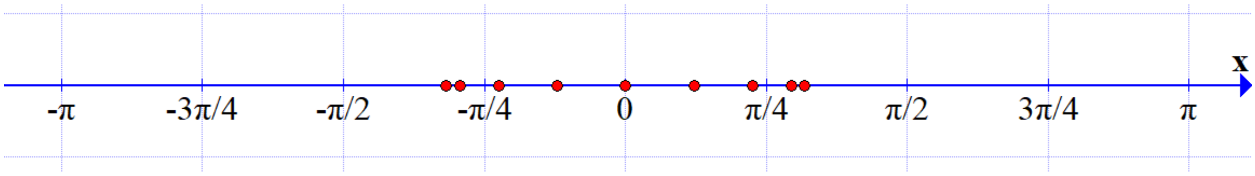


How do the numbers in this interval get transformed when we take the sine of them? In other words was is the transformational effect of $f(x) = \sin x$?

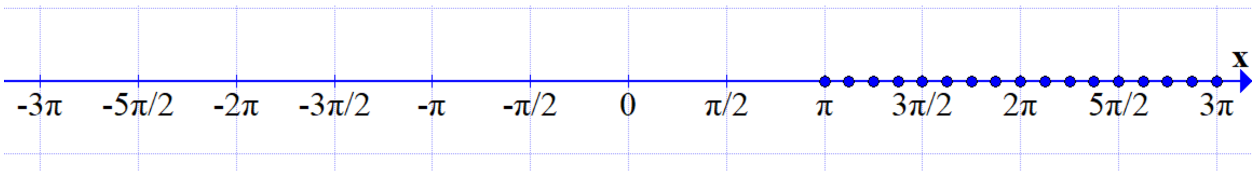
Answer:



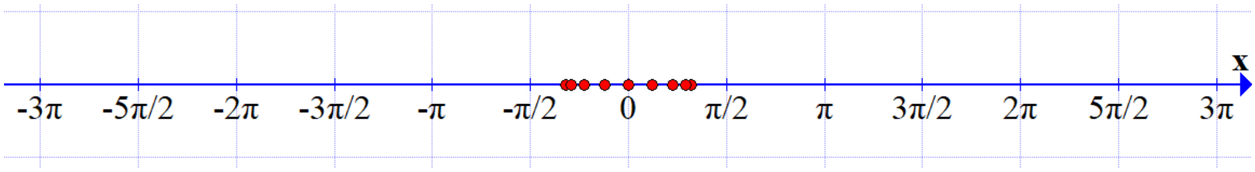
Zooming in we see



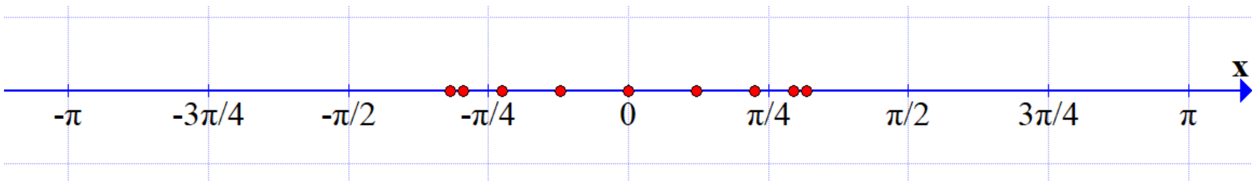
What if we take an interval of π to 2π, as shown below:



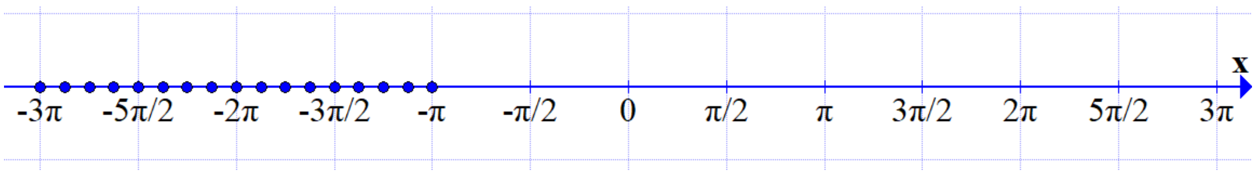
Transforming these numbers according to $f(x) = \sin x$ gives

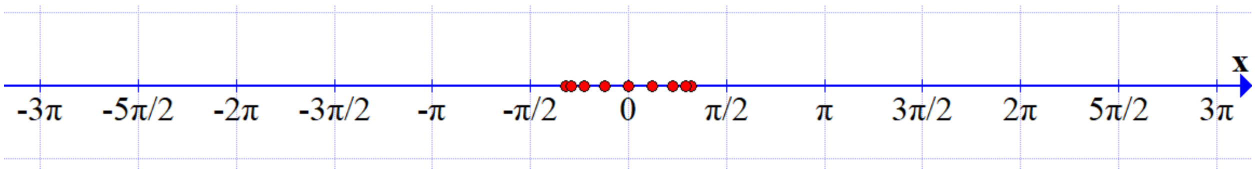


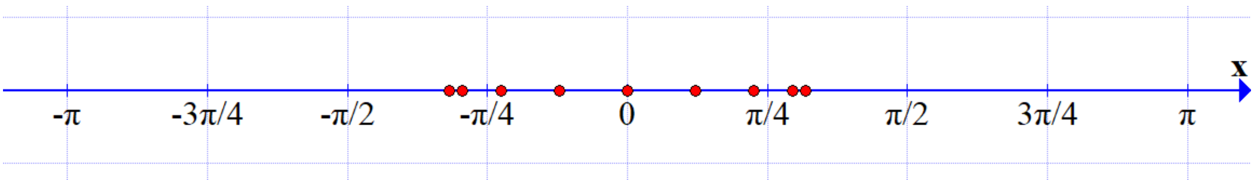
And zooming in we see



But this is the same distribution as the one we got we when used the interval −π to π. What about using the interval −3π to −π? Here we get the following sequence of diagrams:

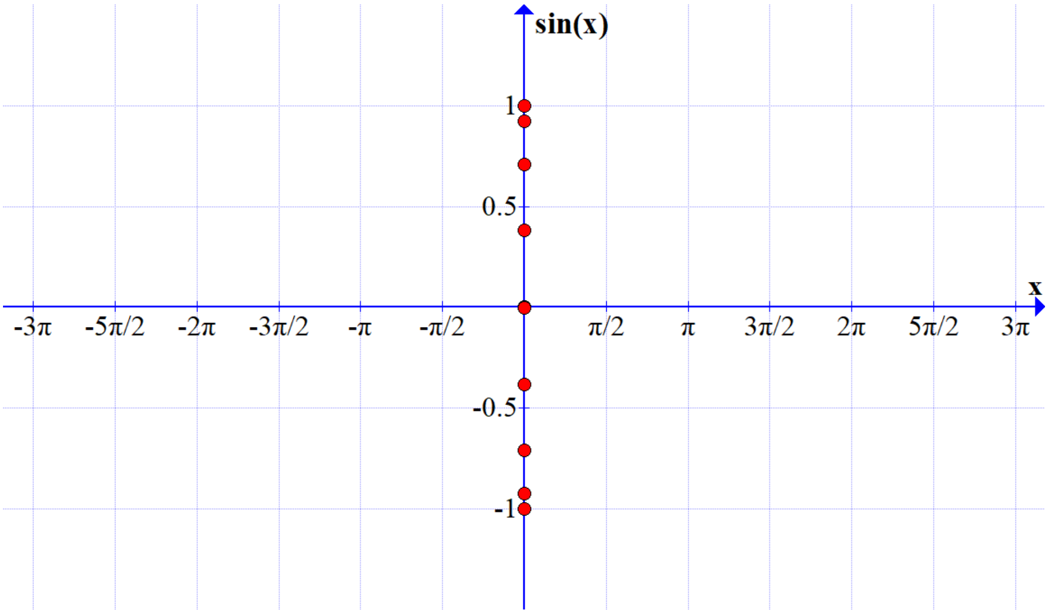




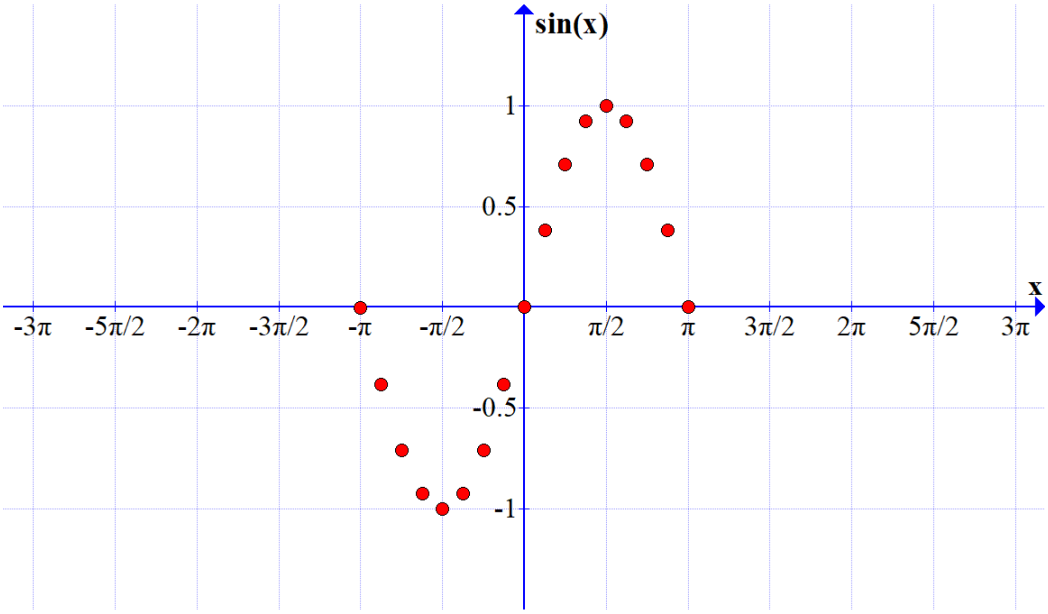


So it seems like whatever interval of numbers we take from the number line (provided the interval covers the same distance), the numbers always get transformed in the same way.

As before, this is not the easiest way to visualise this transformation, so we will repeat the process of illustrating the transformed numbers on a vertical axis …

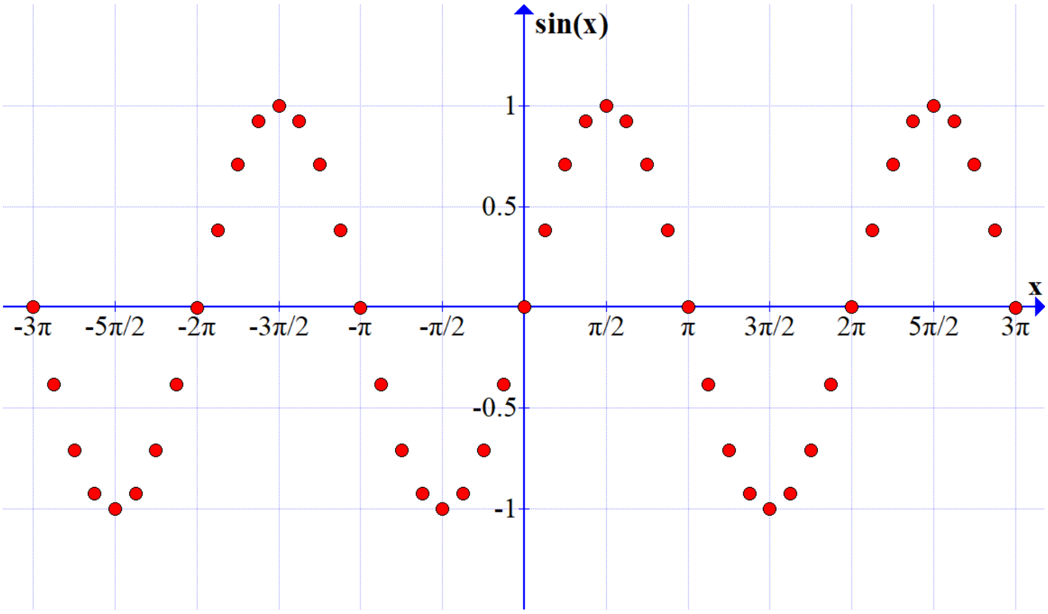


… and then placing each transformed number over its original number:

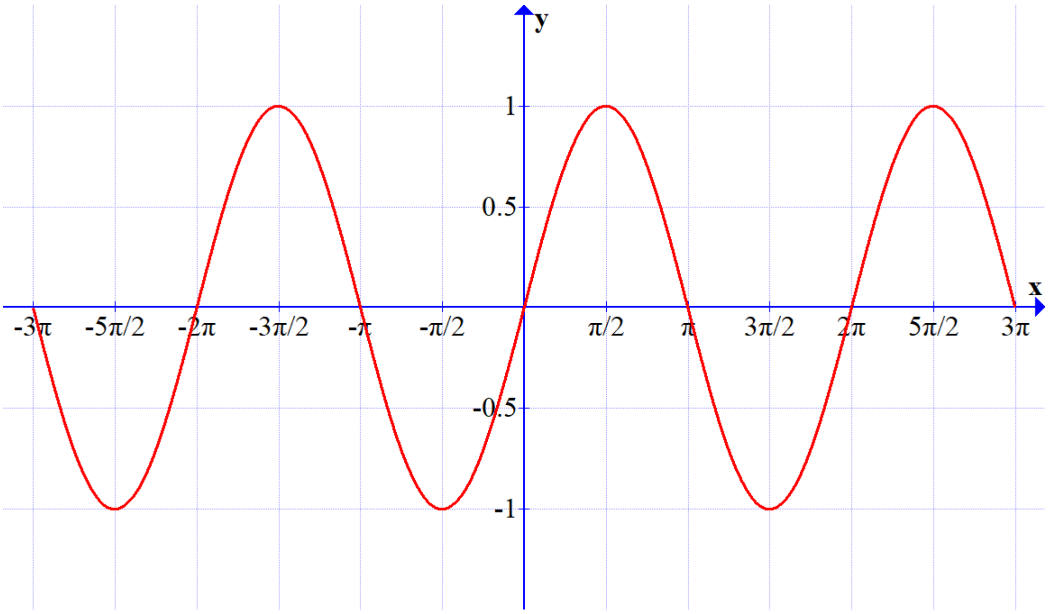


We now see why it is important to do this last step. Illustrating the transformed numbers vertically only allows us to see half of the transformed numbers, because some of these numbers occur twice and therefore overlap each other on the vertical axis.

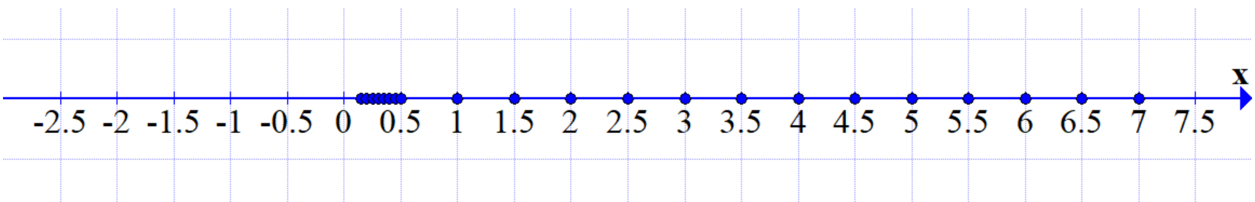
Repeating this approach for the x numbers in the intervals -3π to $-\pi$ and π to 3π we get

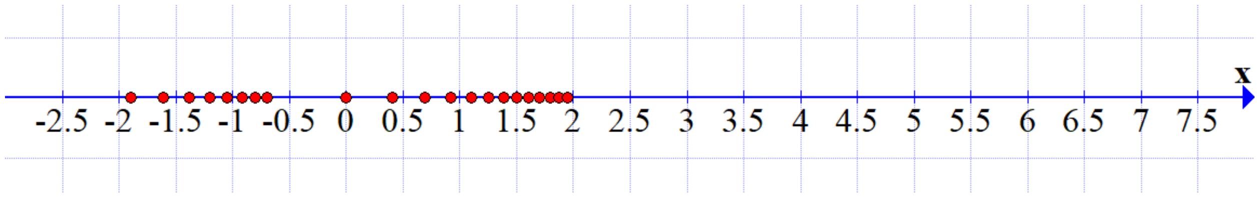


So it seems that, although we get the same distribution of numbers in any given 2π interval, this diagram shows us the transformation of the number line when we transform according to $f(x) = \sin x$. This transformation is the usual sine curve:

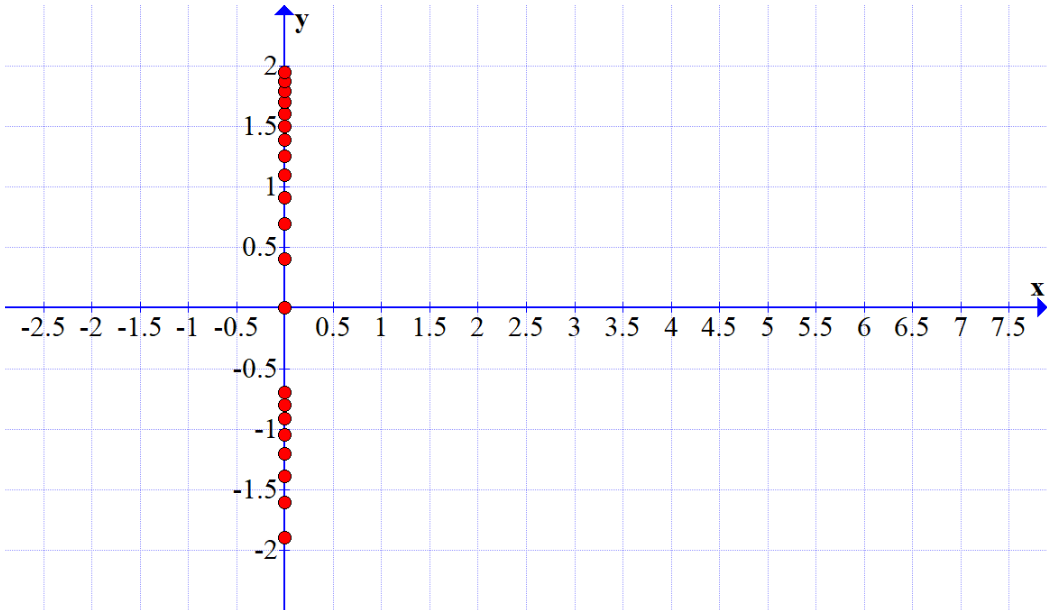


Similarly for $f(x) = \ln x$ we get the following sequence of diagrams, from the untransformed number line (with representative numbers shown by the blue dots) to the transformed number line (with representative numbers shown by the red dots):

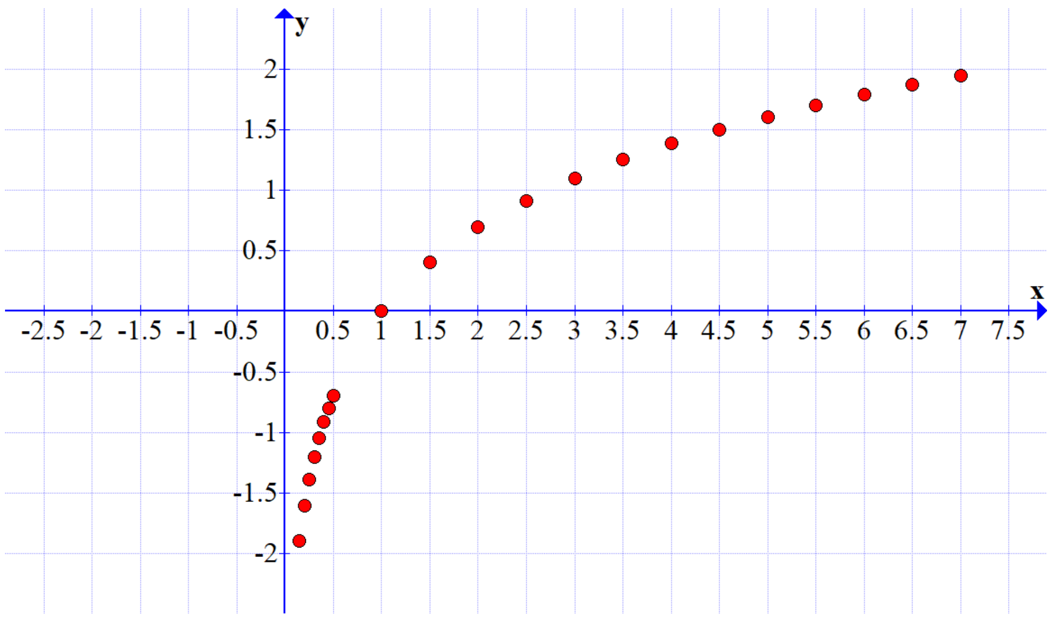




Representing the transformed numbers vertically:



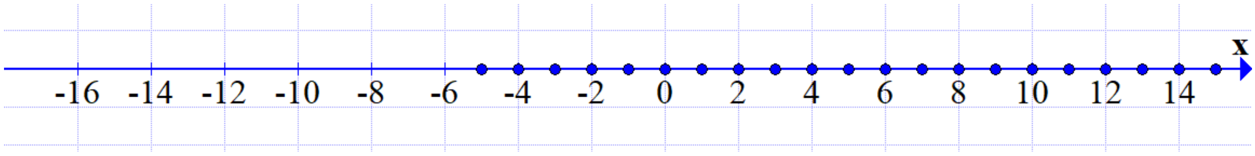
Aligning each transformed numbers over its untransformed number:

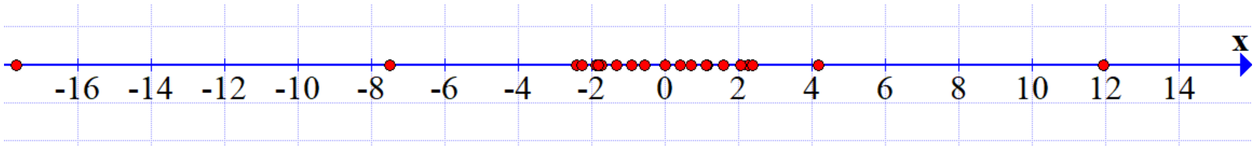


which we can now see to be the usual $\ln x$ function.

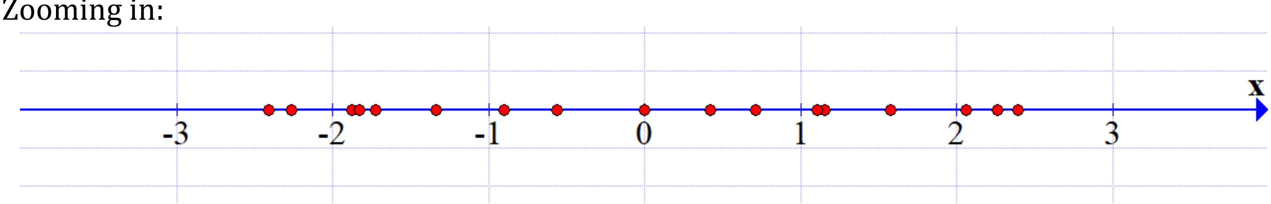
This representation of the distribution of values of a function aplies to any function. What this means is that it doesn't matter how complicated the functions becomes. It still represents a transformation of the number line.

For example, the following sequence of diagrams represents the transformation of the number line (with representative numbers shown by the blue dots) to the transformed number line (with representative numbers shown by the red dots) for the function $f(x) = 10e^{-0.2x} \sin 3x$:

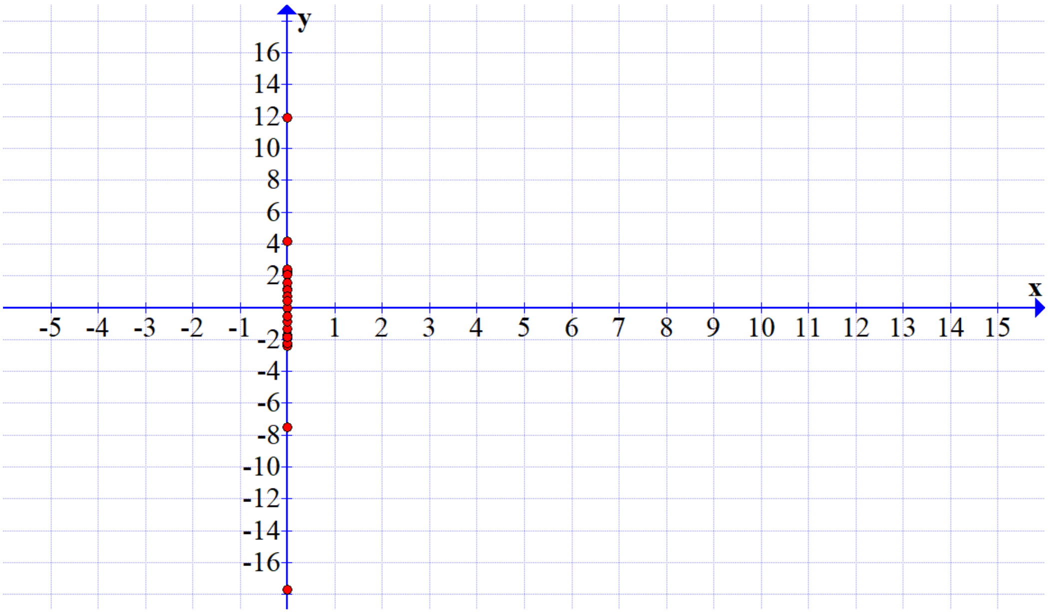




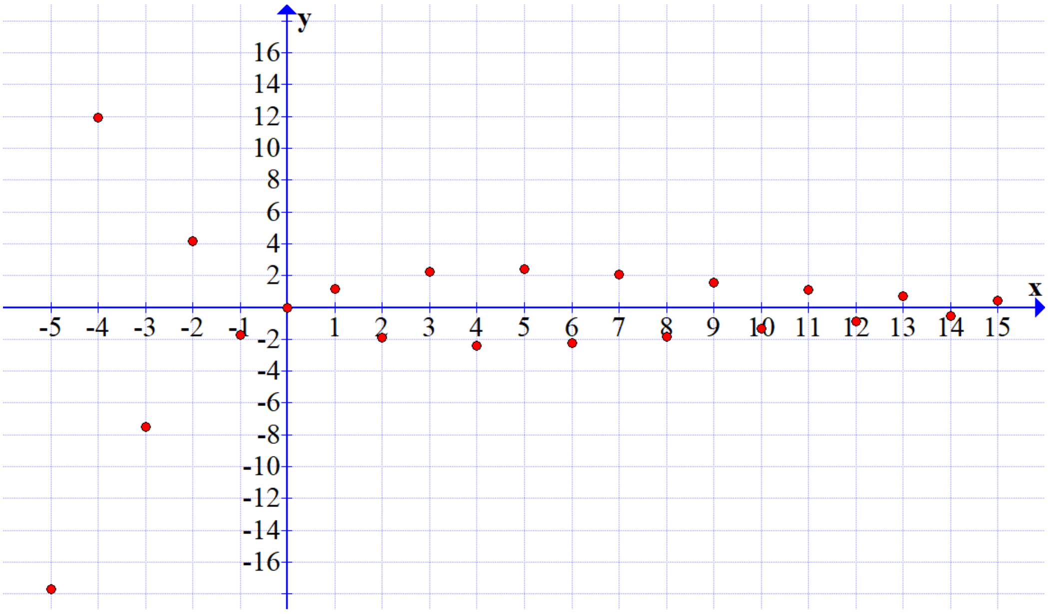
Zooming in:



Representing the transformed numbers vertically we have:

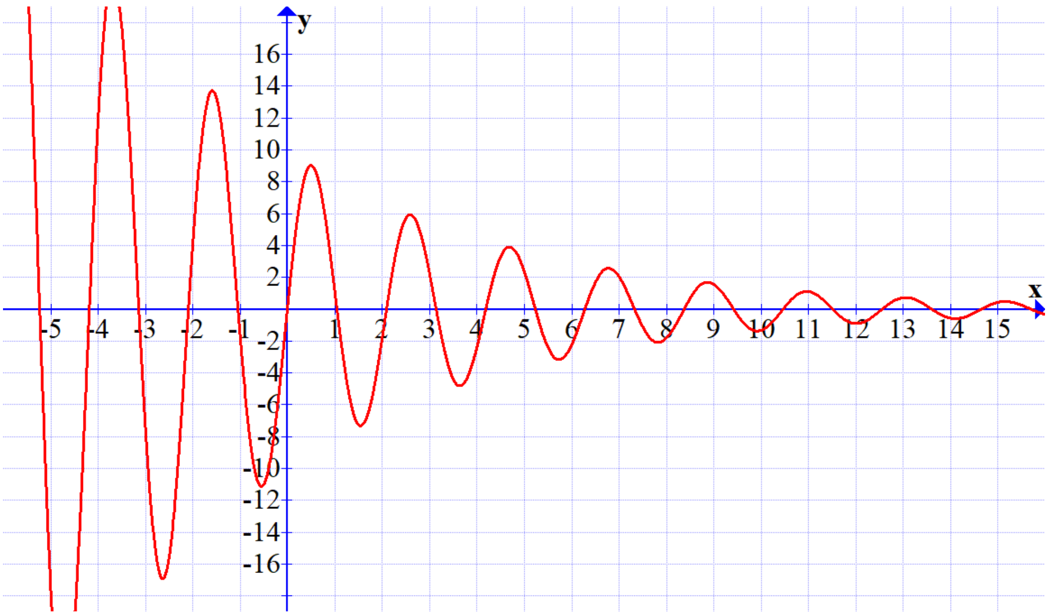


Aligning each transformed numbers over its untransformed number (i.e. the *x*-axis) we have:



In this case it is difficult to see the correct distribution of the dots. From my perspective the distrobution ooks like the body of a fish, with a small head towards the positive *x*-axis and a large tail towards the negative *x*-axis. In this case we would need to plot many more point in order to see the sequential order in which the data distributes from $-\infty$ to $+\infty$.

Ultimately the complete transformation is seen to be:

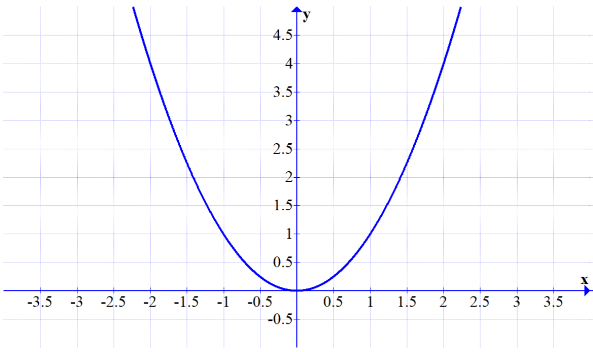


Having discussed the fact that functions act as transformations of the number line the next section deals with how functions themselves can be transformed.

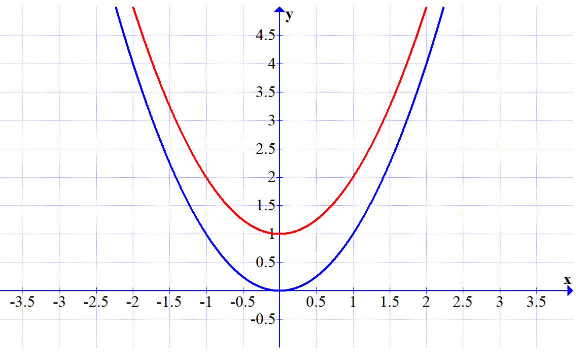
1.2 Describing the behaviour of *f*(*x*) under certain transformations

1.2.1 Transforming f(x) to f(x) ± k, where k is a constant

Consider the function $f(x) = x^2$. What would $f(x) + 1$ look like? How does $f(x) + 1$ change $f(x)$? Looking at the graphs below we can see the effect of adding 1 to x^2.



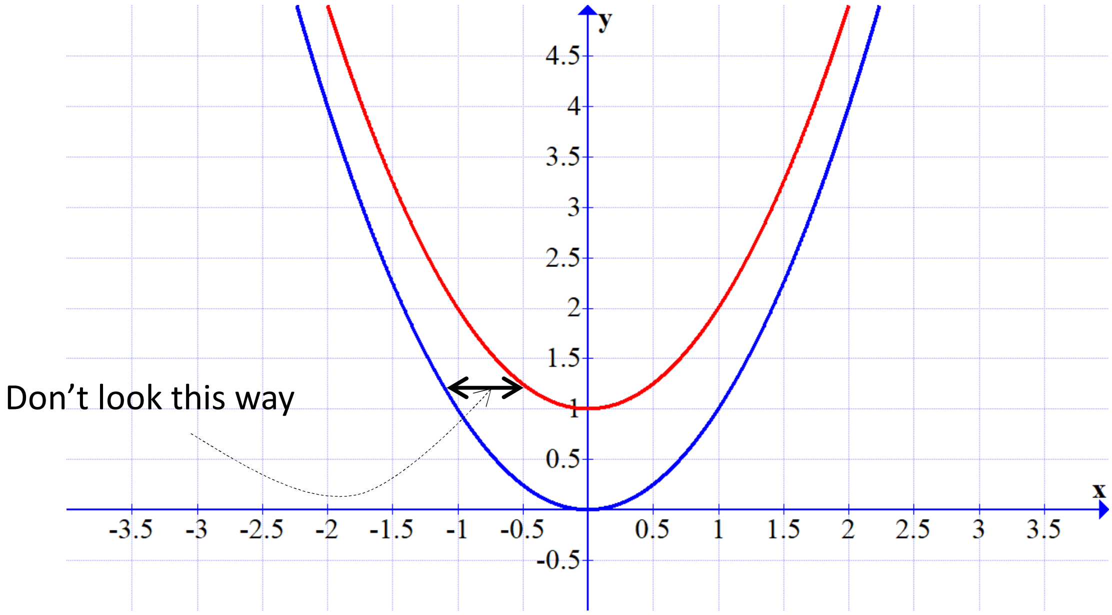
$f(x) = x^2$

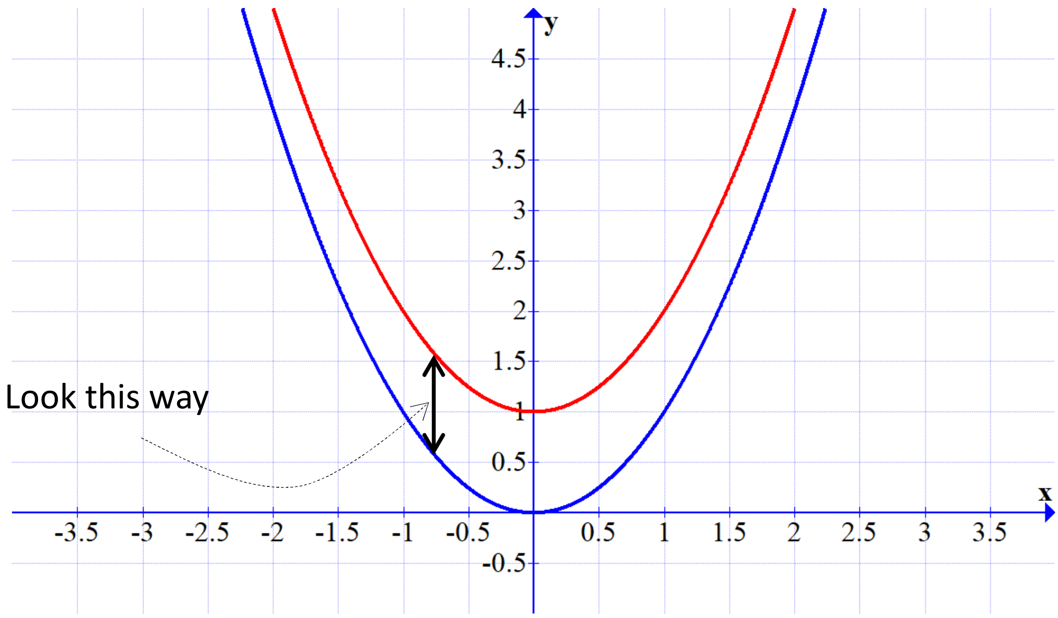


$f(x) + 1 = x^2 + 1$ (in red)

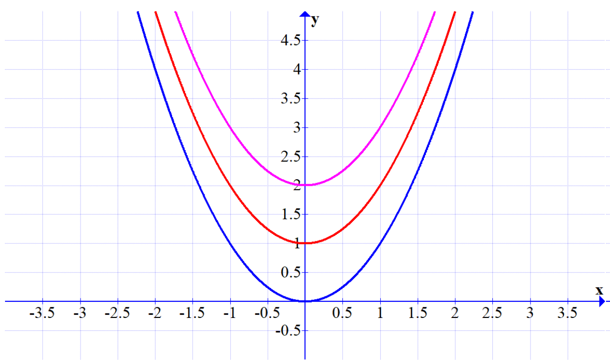
Looking at the difference between the two curves we see that $x^2 + 1$ has shifted vertically upwards compared to x^2. This is the only difference between the two curves. In other words there is no horizontal shifting of x^2 nor any stretching or compressing. It is important to know this since other changes to $f(x)$ will result in stretching or compression.

Now, you might say that $x^2 + 1$ does stretch, or open up, x^2 as we move higher up the curve. But this is just a visual effect cause by looking horizontally across the curves. Looking horizontally across the curves means that you are comparing different x values. We want to compare different $f(x)$ values, and this means looking vertically across the curves. Doing this we see that, for $f(x) + 1$, every value of $f(x) + 1$ has shifted upwards by 1 unit.

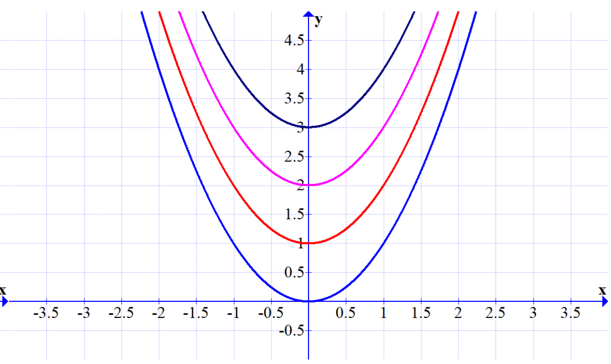




What happens if we do $f(x) + 2$ or $f(x) + 3$? These transformation are shown in the graphs below.



$f(x) + 2 = x^2 + 2$ (in pink)



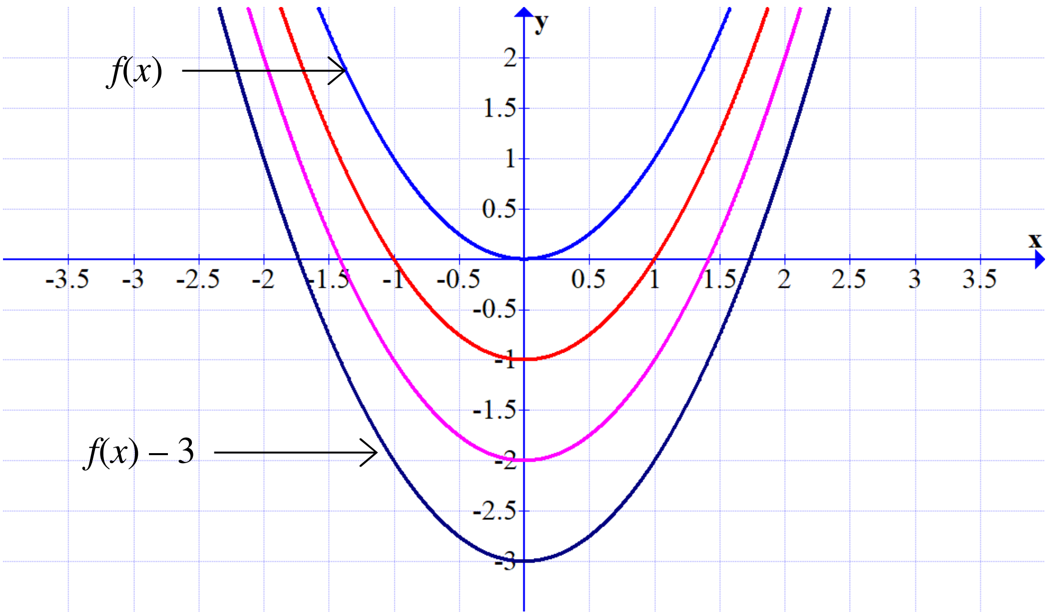
$f(x) + 3 = x^2 + 3$ (in dark blue)

Yet again each curve has moved up 1 unit from its previous position. So, starting with $f(x) = x^2$:

$f(x) + 1$ moves $f(x)$ vertically upward by 1 unit,	$f(x) + 2$ moves $f(x)$ vertically upward by 2 units,
$f(x) + 3$ moves $f(x)$ vertically upward by 3 units,	$f(x) + 4$ moves $f(x)$ vertically upward by 4 units,

etc.

What will happens if we now do $f(x) - 1$, $f(x) - 2$, $f(x) - 3$, ... ? Well, if adding a constant to $f(x)$ translates the curve vertically upwards, then it might seem logical that subtracting a constant from the function translates the curve vertically downwards. This is indeed the case, as we can see in the sequence of graphs below:



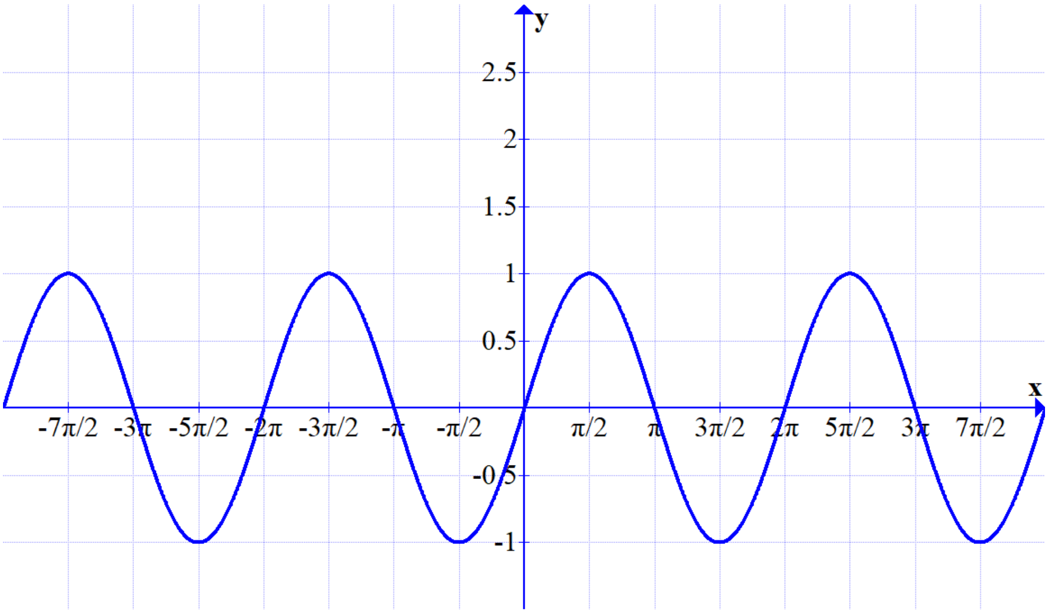
Again, we started with $f(x) = x^2$:

$f(x) - 1$ moves $f(x)$ vertically downward by 1 unit,	$f(x) - 2$ moves $f(x)$ vertically downward by 2 units,
$f(x) - 3$ moves $f(x)$ vertically downward by 3 units,	$f(x) - 4$ moves $f(x)$ vertically downward by 4 units,

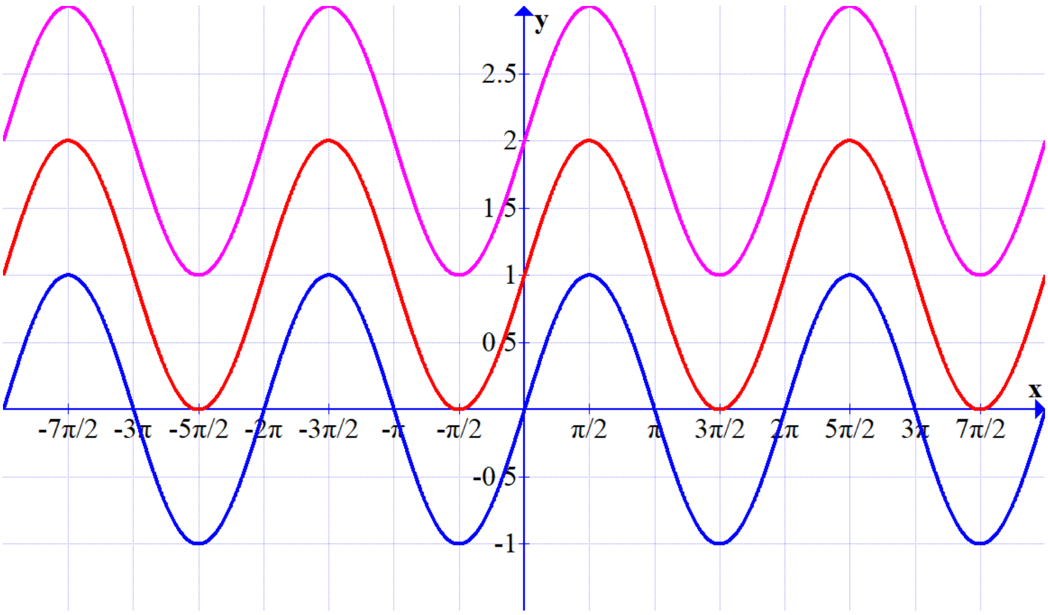
etc.

This property of vertical upwards or downwards movements applies to all functions, not just $f(x) = x^2$.

For example, the graph of $y = \sin(x)$ is

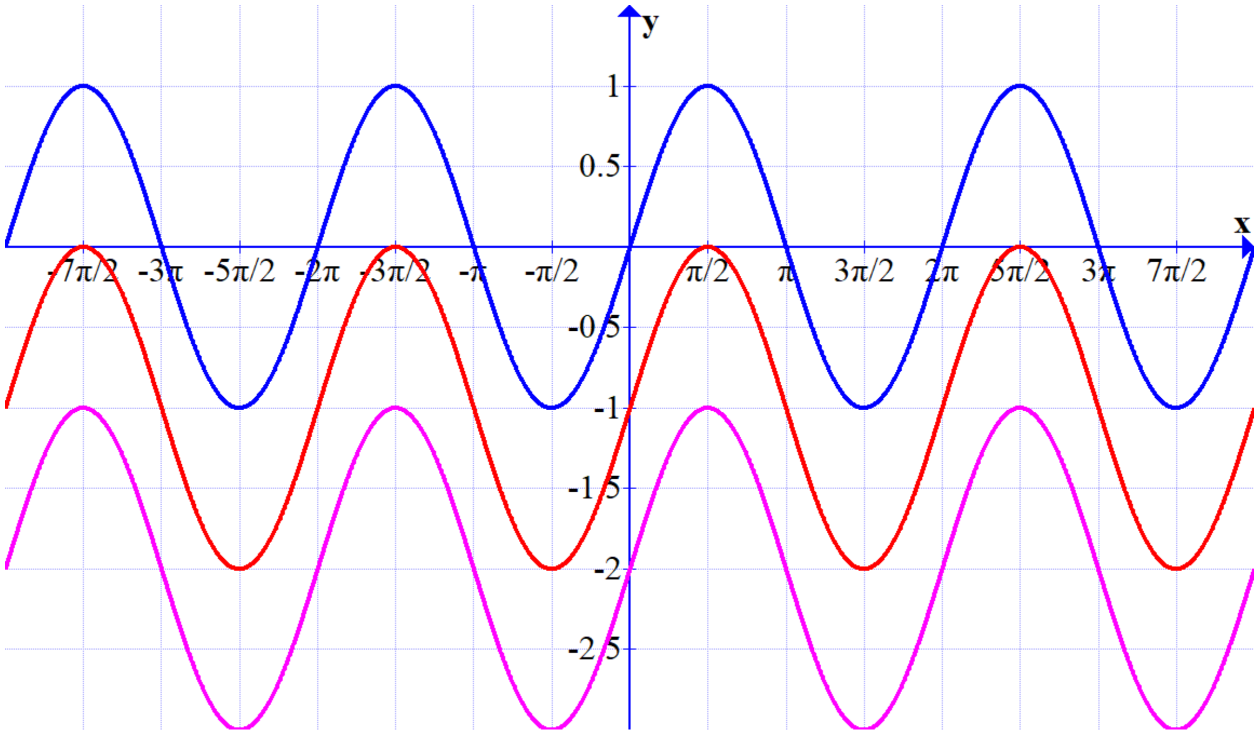


Performing the transformations $y = \sin(x) + 1$ and $y = \sin(x) + 2$ produces the upward translations of 1 and 2 units respectively, as seen in red and pink in the graph below:



Again note that nothing else about the shape of the curve changes: no stretching, no compression, no rotation, etc…

Similarly for a downward translation we have, for $y = \sin(x) - 1$ and $y = \sin(x) - 2$:



So we can now say that, in general, given a function $f(x)$,

the effect of doing $f(x) \pm k$ is to translate $f(x)$

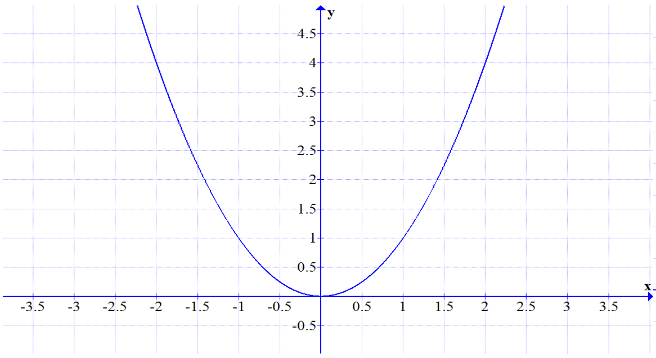
vertically upwards or downwards by k units.

It doesn't matter how complicated $f(x)$ is, the effect of doing $f(x) \pm k$ is always to translate $f(x)$ vertically by k units.

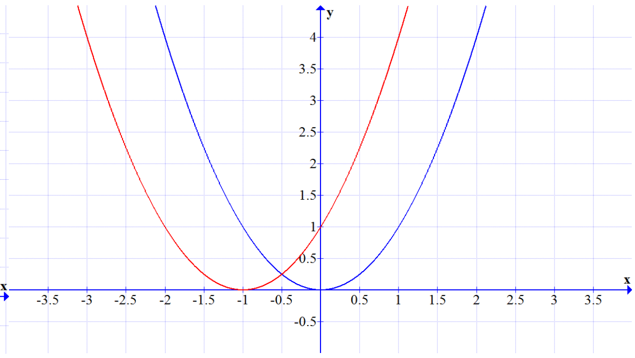
1.2.2 Transforming f(x) to f(x ± k), where k is a constant

We have seen the effect on a function of adding a constant to the function. What is the effect on the function of adding a constant to the independent variable x?

Consider again the function $f(x) = x^2$. What would $f(x+1)$ look like? How does $f(x+1)$ change $f(x)$? Looking at the graphs below we can see the effect of this addition to the variable x.



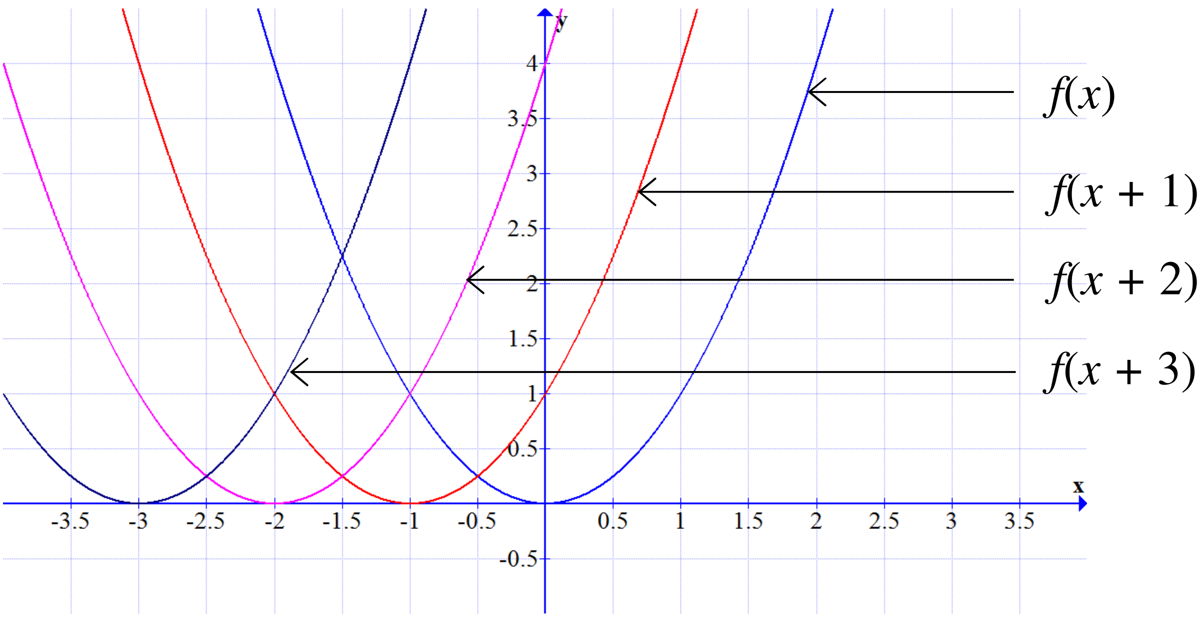
$f(x) = x^2$



$f(x+1) = (x + 1)^2$ (in red)

Looking at the difference between the two curves we see that $(x + 1)^2$ has translated lefward compared to x^2. This is the only difference between the two curves. In other words there is no vertical shifting of x^2 nor any stretching or compressing.

What happens if we do $f(x+2)$ or $f(x+3)$? These transformation are shown in the graphs below.



Yet again each curve has moved 1 unit to the left from its previous position. So, starting with $f(x) = x^2$:

$f(x + 1)$ translates $f(x)$ leftward by 1 unit,

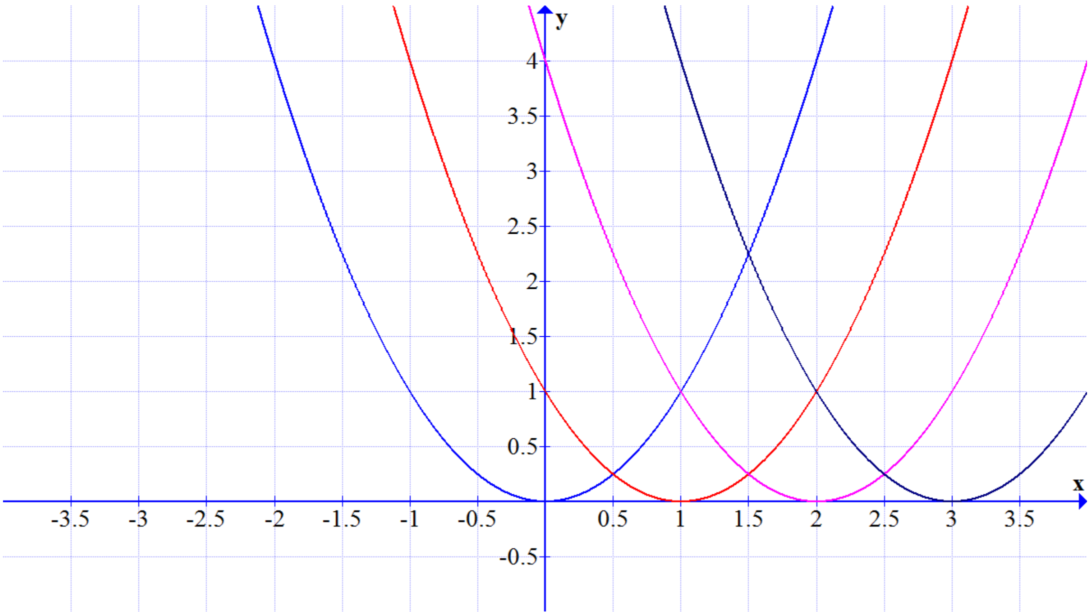
$f(x + 2)$ translates $f(x)$ leftward by 2 units,

$f(x + 3)$ translates $f(x)$ leftward by 3 units,

$f(x + 4)$ translates $f(x)$ leftward by 4 unit,

etc.

What will happens if we now do $f(x- 1)$, $f(x-2)$, $f(x-3)$, ... ? Well, if adding a constant to x translates the curve leftward, then it might seem logical that subtracting a constant from x translates the curve rightward. This is indeed the case, as we can see in the sequence of graphs below:



So we can now say that, in general, given a function $f(x)$,

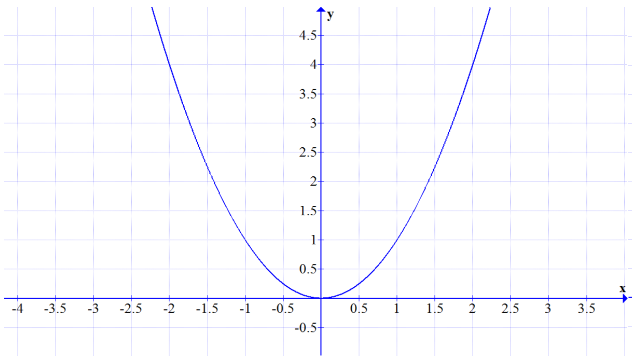
the effect of doing $f(x \pm k)$ is to translate $f(x)$ leftward or rightward by k units.

(Note that in trigonometry this left/right translation is called the phase shift of the trig function)

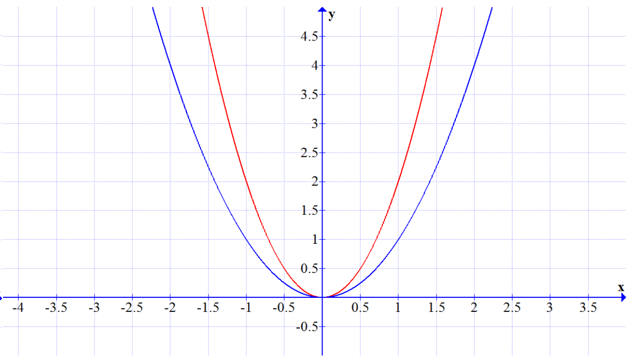
1.2.3 Transforming f(x) to k.f(x), where k is a constant

We have seen the effect on a function of adding constants (whether to the functionr to the variabel x). What is the effect on the function of multiplying it by a constant?

Consider again the function $f(x) = x^2$. What would $2.f(x)$ look like? How does $2.f(x)$ change $f(x)$? Looking at the graphs below we can see the effect of this multiplication:

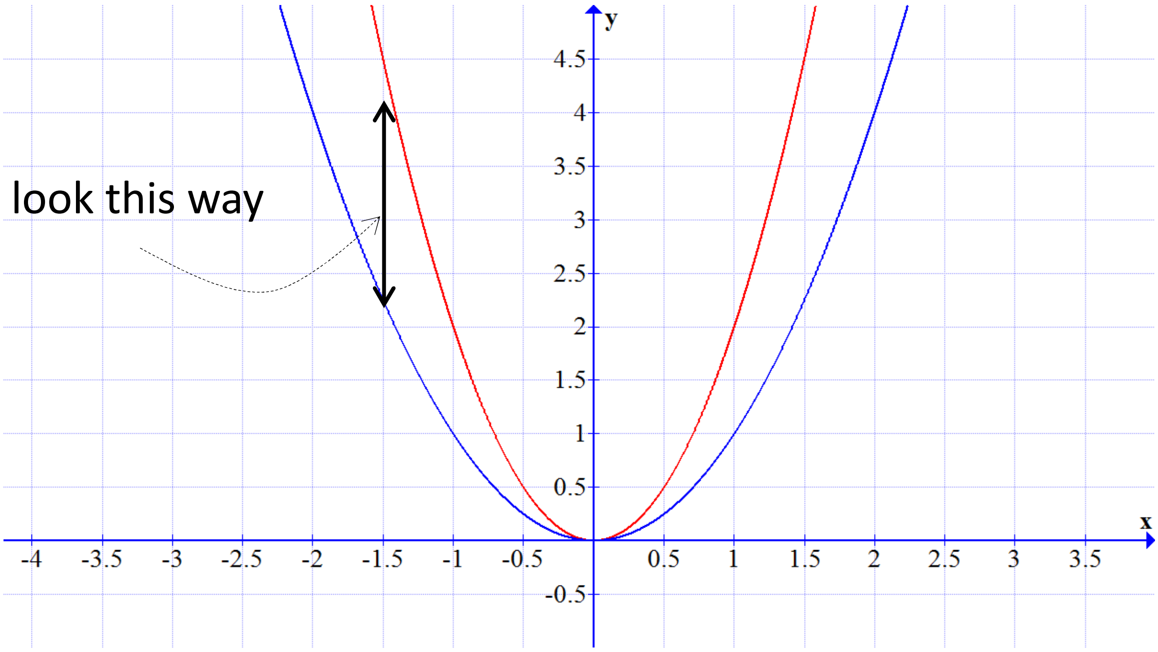


$f(x) = x^2$

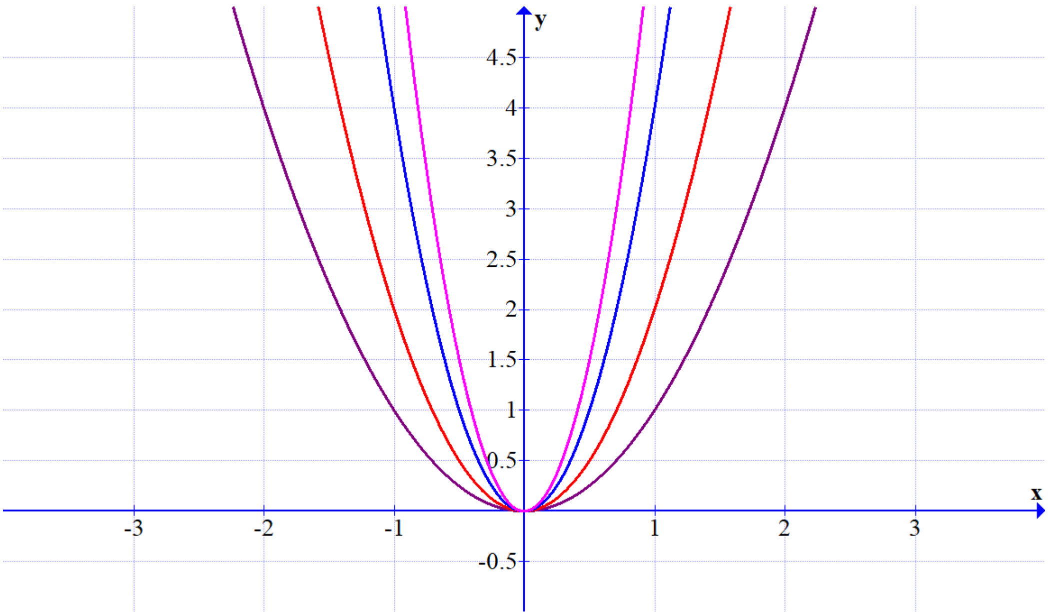


$2.f(x) = 2x^2$ (in red)

Looking at the difference between the two curves it seems that $2x^2$ has been compressed inwards towards the y-axis compared to x^2. This is technically an incorrect view of what has happened. It is just a visual effect cause by looking horizontally across the curves. We want to compare different $f(x)$ values, and this means looking vertically across the curves. In doing so we find that every value of $2.f(x)$ has streched vertically upwards (away from the x-axis) by a factor of 2 compared to $f(x)$.

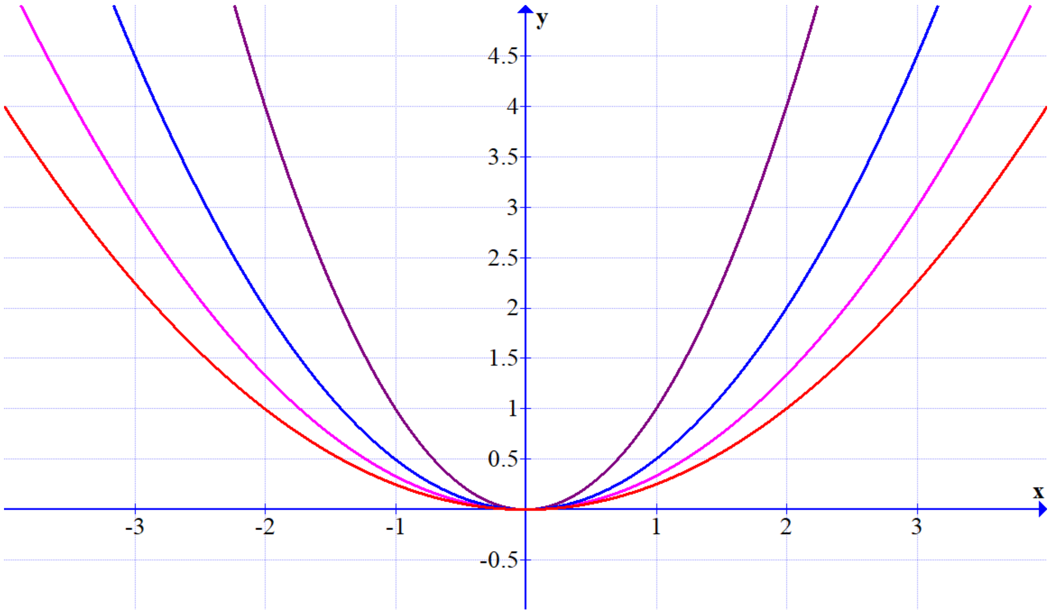


What happens if we do 4.*f*(*x*) or 6.*f*(*x*)? These transformation are shown in the graphs below.



*4.f(x) = 4*x^2 *(in blue), 6.f(x) = 6*x^2 *(in pink)*

What will happens if we now do ½.*f*(*x*), $^1/_3$.*f*(*x*), ¼.*f*(*x*), ... ? Well, if multiplying *f*(*x*) by a number bigger than 1 stretches *f*(*x*) upwards then it might seem logical that multiplying *f*(*x*) by a number between 0 and 1 stretches *f*(*x*) downwards (towards the *x*-axis). This is indeed the case, as we can see in the sequence of graphs below:



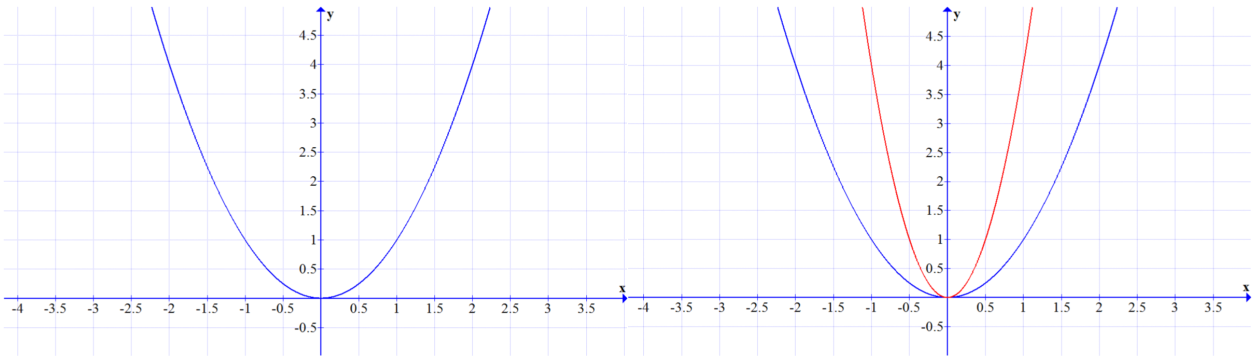
So we can now say that, in general, given a function $f(x)$,

the effect of doing $k.f(x)$ is to stretch or compress $f(x)$ by k units away from or towards the x-axis.

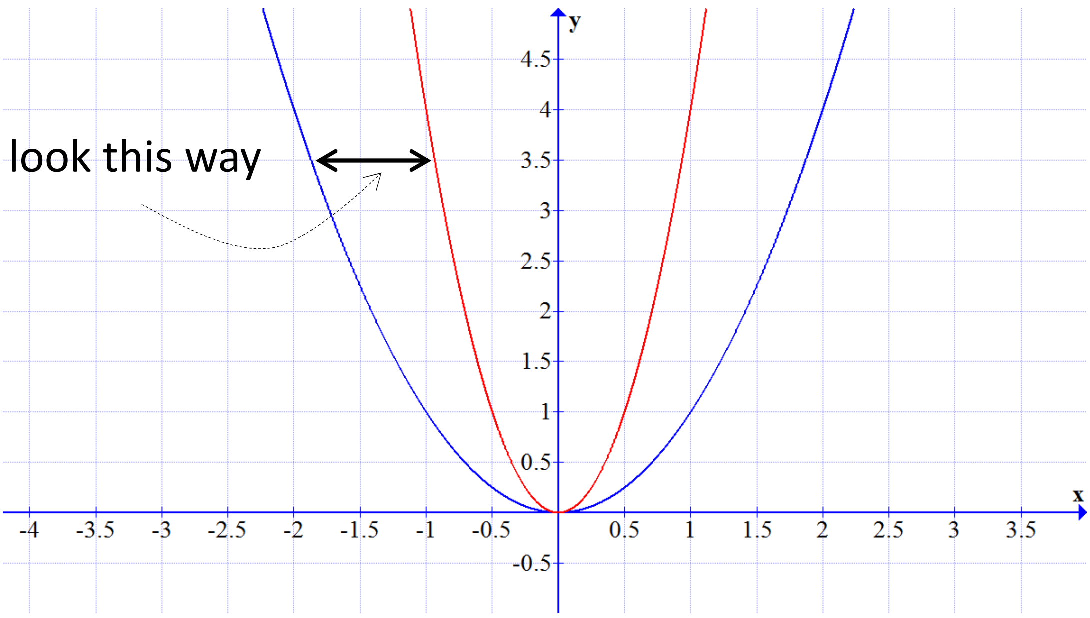
1.2.4 Transforming f(x) to f(kx), where k is a constant

We have seen the effect on a function of multiplying the function by a constant? What is the effect on the function if we multiply the *x* variable by a constant?

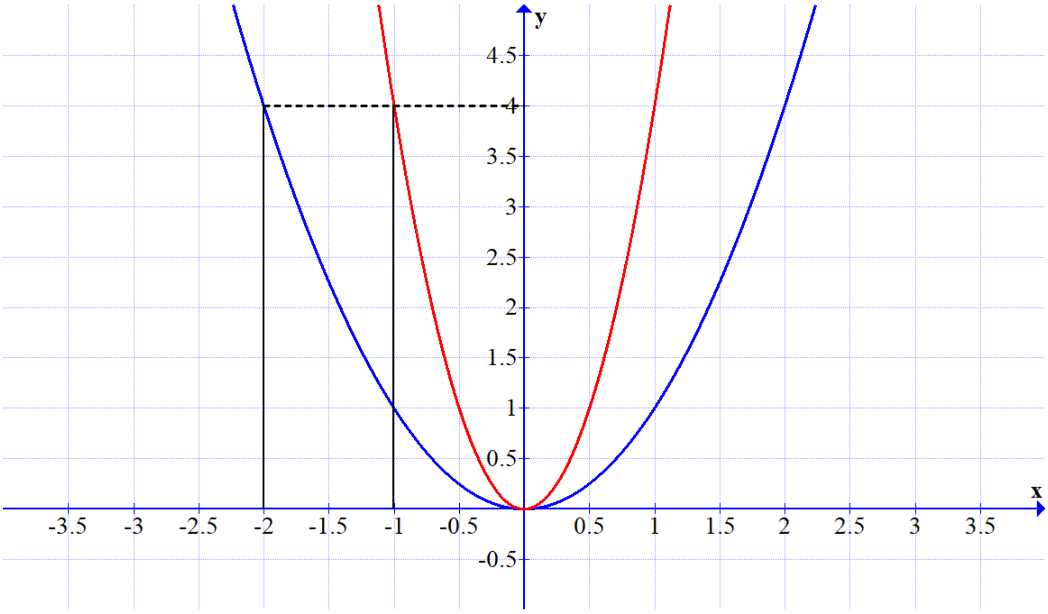
Consider again the function $f(x) = x^2$. What would $f(2x)$ look like? How does $f(2x)$ change $f(x)$? Looking at the graphs below we can see the effect of this multiplication:



The effect of such a transformation is more tricky to see because it looks similar to that of $f(x) = x^2$. However there is a difference which can be explained by the direction in which $f(x)$ changes. Looking at the difference between the two curves it seems that $(2x)^2$ has been compressed inwards towards the *y*-axis compared to x^2. This is technically correct, but let us be more specific in what is actually happening. What we need to do is to compare the value of $f(x)$ evaluated at different *x* values. This means looking horizontally across the curves.



In doing so we find that every value of $f(x)$ has been compressed horizontally inwards (towards the y-axis) as if it had been evaluated at $x/2$. In other words, what was $f(1)$ (the value of $f(x)$ and $x = 1$) is now $f(½)$ (the value of $f(2x)$ at $x= ½$), what was $f(2)$ (the value of $f(x)$ and $x = 2$) is now $f(1)$, (the value of $f(2x)$ at $x= 1$), etc.



$f(x) = x^2$ (in blue), $f(2x) = (2x)^2$ (in red)

What will happens if we now do $f(½x)$, $f(^1/_3x)$, $f(¼x)$, ... ? Well, if multiplying $f(x)$ by a number bigger than 1 compresses $f(x)$ towards the y-axis then it might seem logical that multiplying $f(x)$ by a number between 0 and 1 stretches $f(x)$ away from the y-axis. This is left as an exercise for you to confirm.

So we can now say that, in general, given a function $f(x)$,

the effect of doing $f(kx)$ is to stretch or compress $f(x)$ by k units away from or towards the y-axis / vertical axis.

1.2.5 Transforming f(x) to |f(x)|

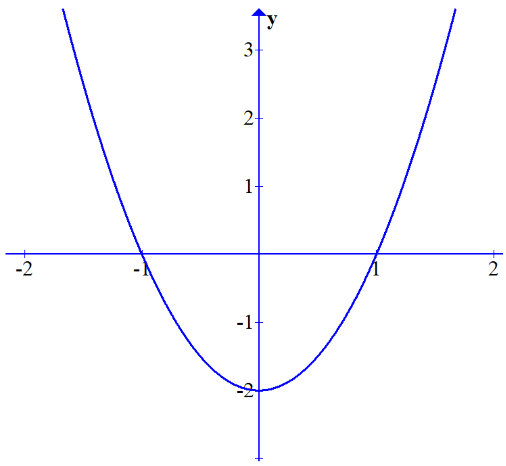
Here we will analyse the effect on a function of taking the modulus of the function. Therefore, consider the function $f(x) = 2x^2 - 2$. What would $|f(x)|$ look like? How does $|f(x)|$ change $f(x)$?

Before we plot $|f(x)|$, let us go over what the modulus operator does. In general the modulus operator does the following to a variable z:

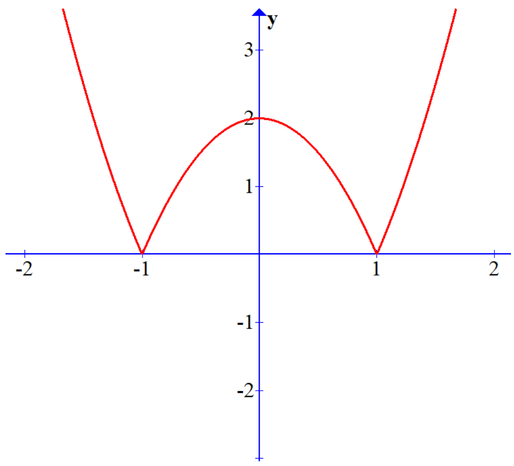
$$|z| = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z < 0 \end{cases}$$

In other words, if z is a positive number, then $|z|$ keeps z positive. If z is a negative number, then $|z|$ makes z positive. So, whatever the value of z, $|z|$ will always make z positive. In the case of $f(x)$ whatever the value of $f(x)$, $|f(x)|$ will always be positive.

So given $f(x) = 2x^2 - 2$, $|f(x)| = |2x^2 - 2|$ would give the result seen below



$f(x) = 2x^2 - 2$,



$|f(x)| = |2x^2 - 2|$

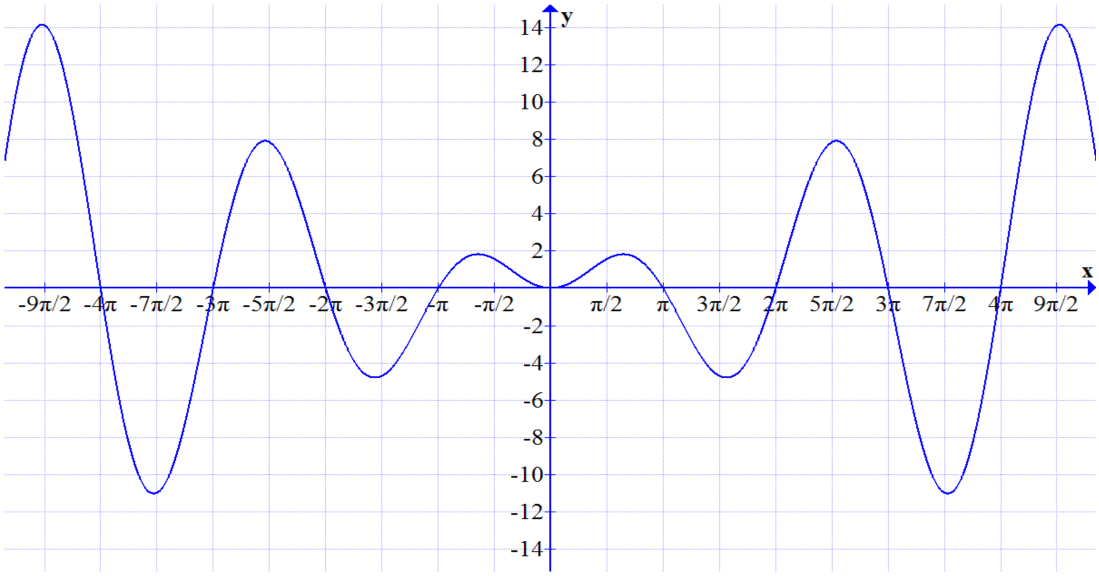
We can therefore see that any part of the curve of $f(x)$ which lies underneath the x-axis (i.e. which is negative) is reflected about the x-axis to lie above the x-axis (i.e. to become positive)

Informally speaking,

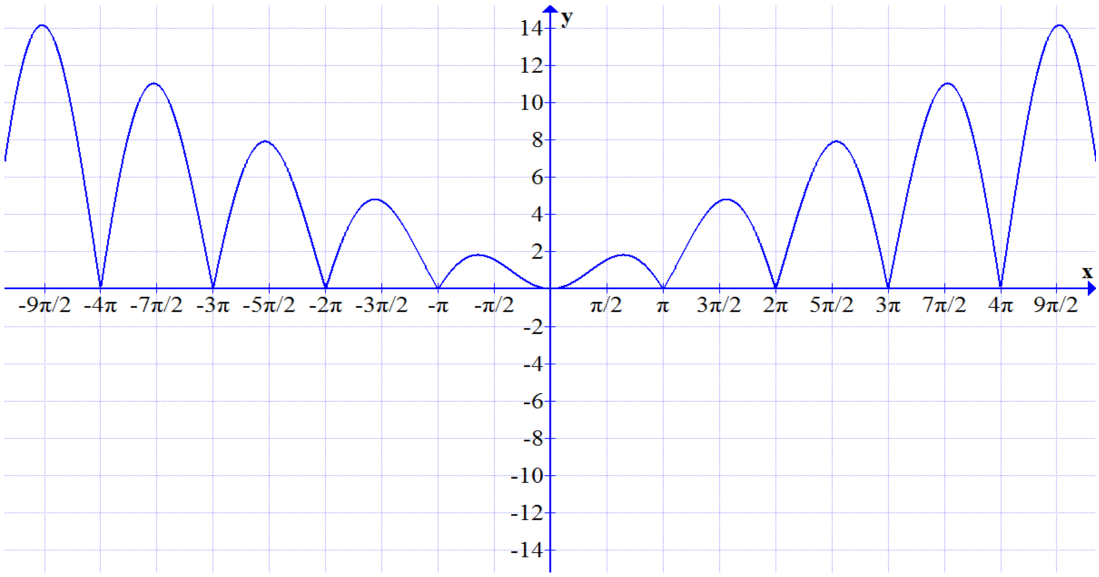
- all the parts of the blue curve which were already positive (i.e. above the x-axis) stay positive;
- all the parts of the blue curve which were negative (i.e. underneath the x-axis) now become positive.

This behaviour of $|f(x)|$ compared to $f(x)$ is always the case whatever the function $f(x)$.

As an example of this let us look at the function $f(x) = x.\sin(x)$, a graph of which is shown here



The curve of $|f(x)| = |x.\sin(x)|$ then looks like



1.2.6 Transforming f(x) to f(|x|)

We have seen the effect on a function of taking the modulus of that function? But what will the effect on $f(x)$ be if we take the modulus of the variable x instead? In other words what will happen if we do $f(|x|)$?

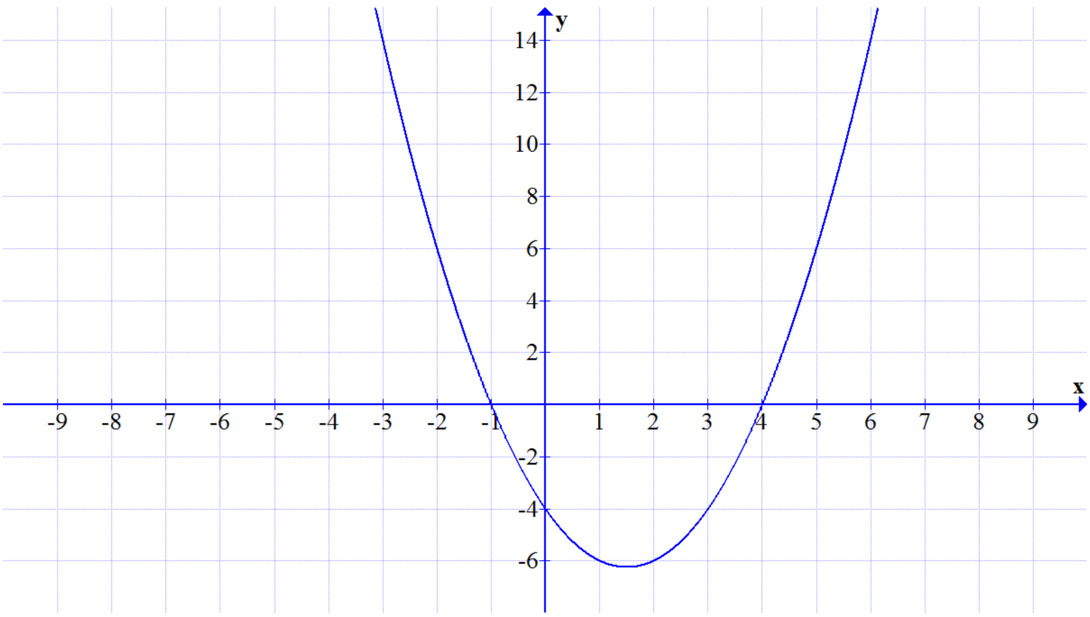
Here we are saying that whatever the value of x, we will always evaluate the function at the positive value of x. In other words,

If we do ...	... we calculate
$f(\|1\|)$	$f(1)$
$f(\|2\|)$	$f(2)$

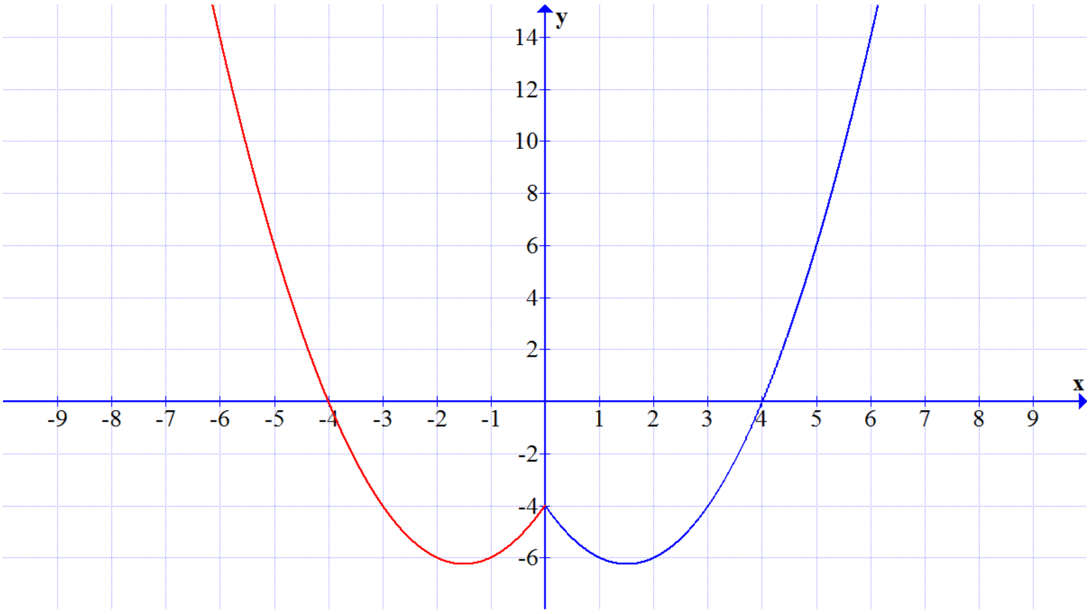
But also,

If we do ...	... we calculate
$f(\|-1\|)$	$f(1)$
$f(\|-2\|)$	$f(2)$

To see the effect of $f(|x|)$ on $f(x)$ consider the function $f(x) = x^2 - 3x - 4$ shown below

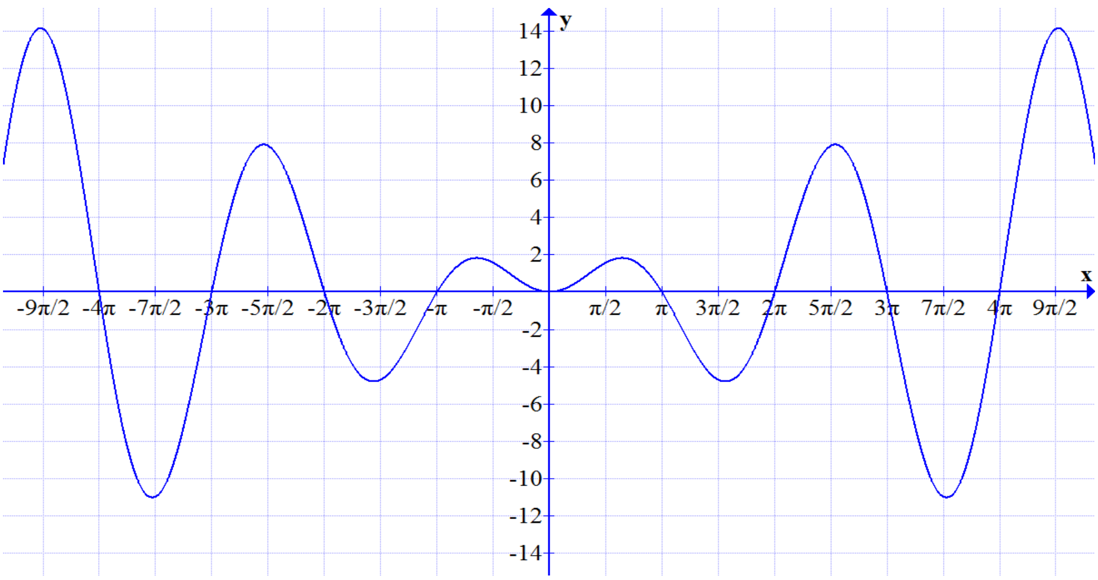


The curve $f(|x|) = |x|^2 - 3|x| - 4$ then looks like

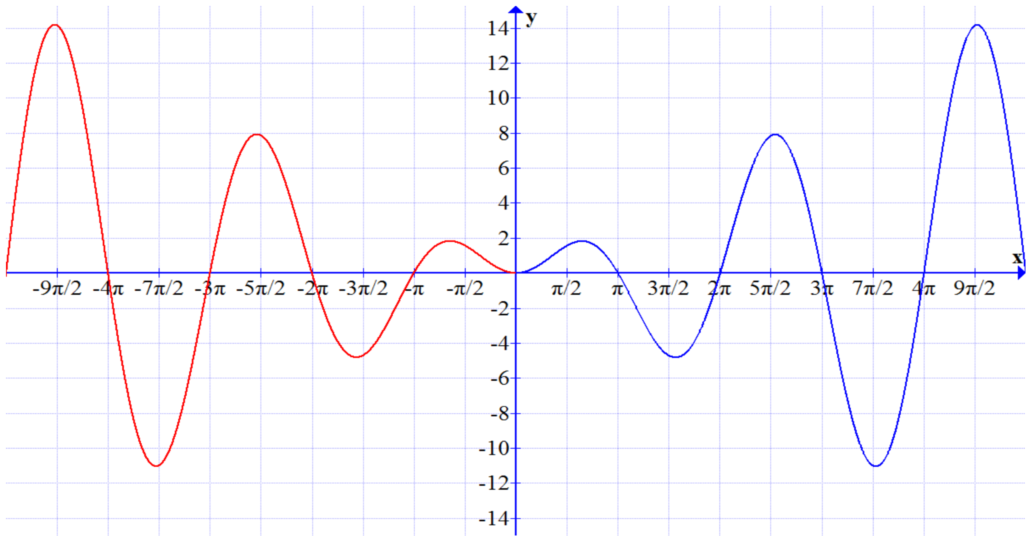


Visually speaking this means that whatever the curve looks like on the positive side of the *x*-axis, it will look exactly the same on the negative side of the *x*-axis. I.e. it will be a reflection about the *y*-axis of that part of the curve on the positive side of the *x*-axis. This is the case whatever the function $f(x)$.

As an example of this let us look again at the function $f(x) = x.\sin(x)$, a graph of which is shown here



Reflecting about the y-axis only that part of $f(x)$ which is on the positive side of the x-axis we get that part shown in red below



In other words the curve of $f(|x|)$ looks exactly the same as the curve of $f(x)$. Why? Because in the original function the part of the curve on the negative side of the x-axis was already a reflection of the part of the curve on the positive side of the x-axis.

1.3 One complete example

Let us now look at an example where all (but one) of the above transformations are done in one go. As such, consider $f(x) = x^2$. What will $3f(2|x| - 1) - 3$ look like? In other words we want to know what $3(2|x| - 1)^2) - 3$ will look like.

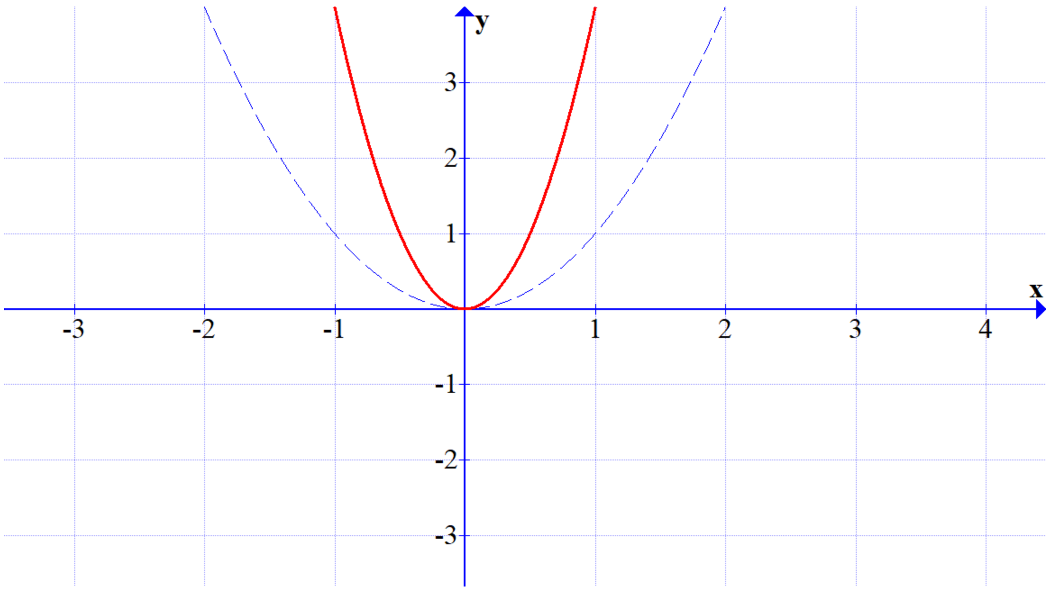
We can analyse this in stages. We know what x^2 looks like. So

1. now consider $(2x)^2$. This will compress x^2 towards the y-axis;
2. now consider $(2x - 1)^2$. This will translate the curve $(2x)^2$ by ½ unit to the right (not 1 unit since we have a term "$2x$" not just a term "x". If necessary go back to the section on the effect of $f(kx)$ to re-read the reason why);
3. now consider $(2|x| - 1)^2$. This will reflect in the y-axis all values of $(2x - 1)^2$ which are on the positive side of the x-axis;
    - Note at this point that the transformation would not have been as obvious if we had decided to follow the sequence $(2x)^2 \rightarrow (2|x|)^2 \rightarrow (2|x| - 1)^2$. Try this sequence to see if you could have predicted the shape of $(2|x| - 1)^2$.

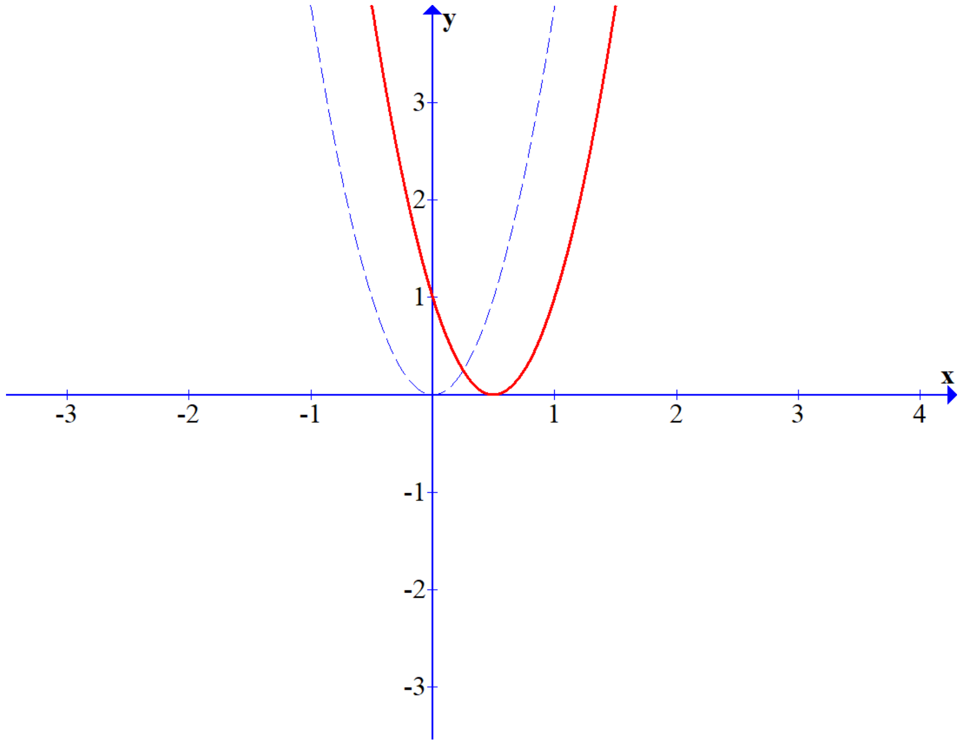
4. now consider $3(2|x| - 1)^2$. This will stretch the curve $(2|x| - 1)^2$ away from the x-axis by a factor of 3;

5. now consider $3(2|x| - 1)^2 - 3$. This will translate the curve $3(2|x| - 1)^2$ by 3 units downward.

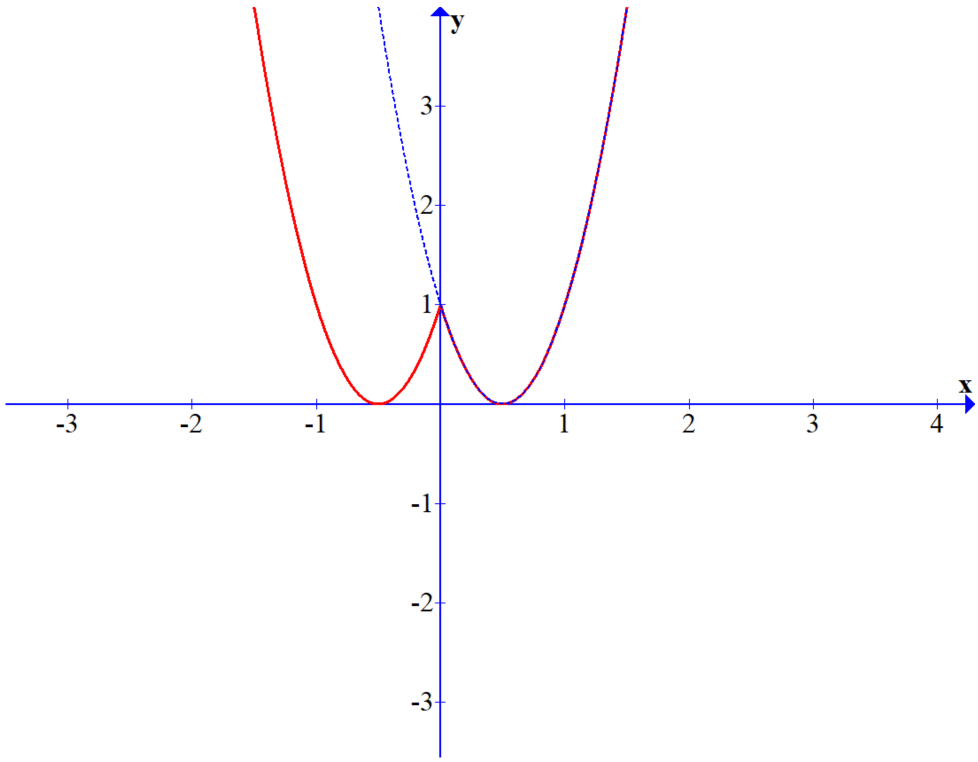
The sequence of transformation described above is shown below where, at each stage the previous function is shown as a blue dashed line and the current transformation is shown as a solid red line



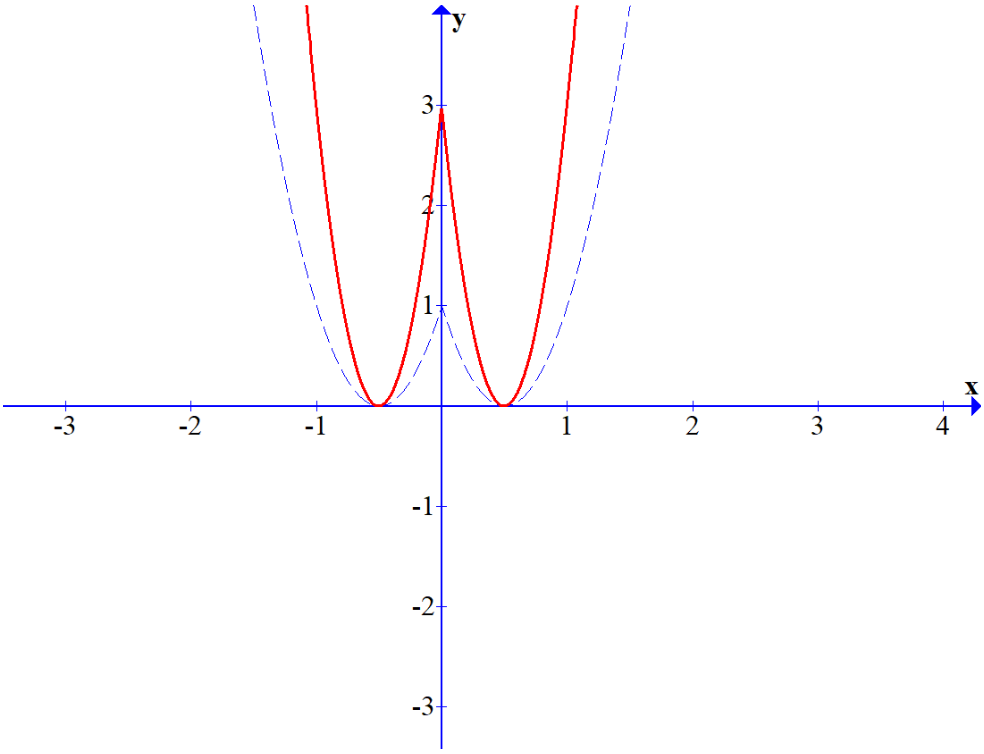
$f(x) = (2x)^2$



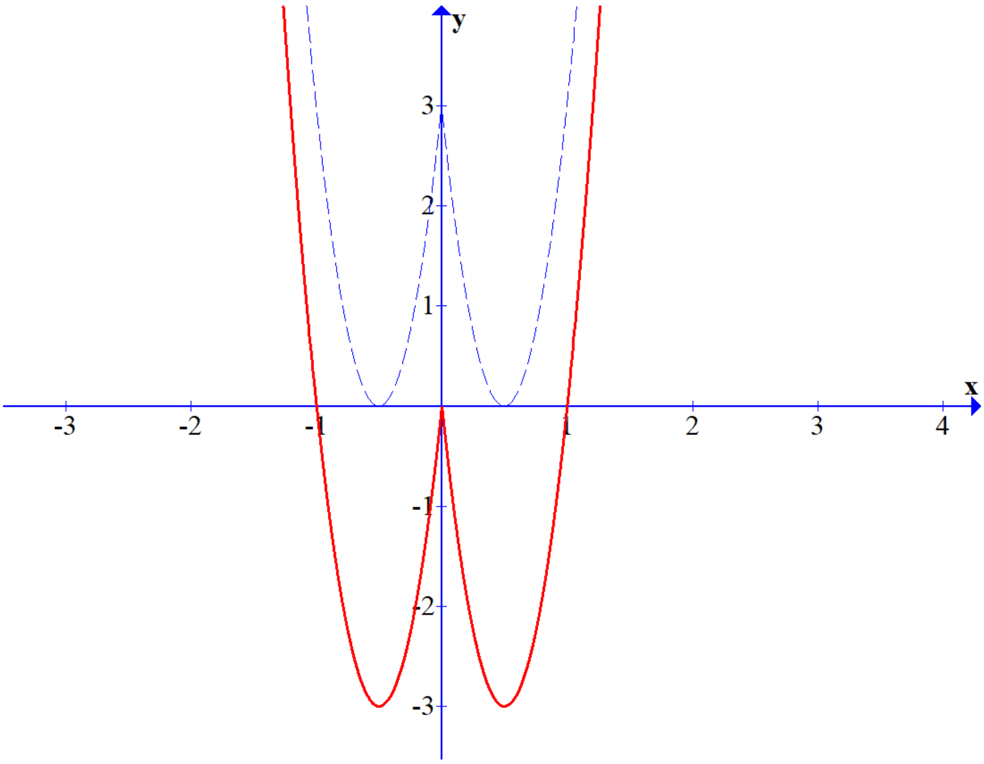
$f(x) = (2x - 1)^2$



$f(x) = (2|x| - 1)^2$



$f(x) = 3(2|x| - 1)^2$



$f(x) = 3(2|x| - 4)^2 - 3$

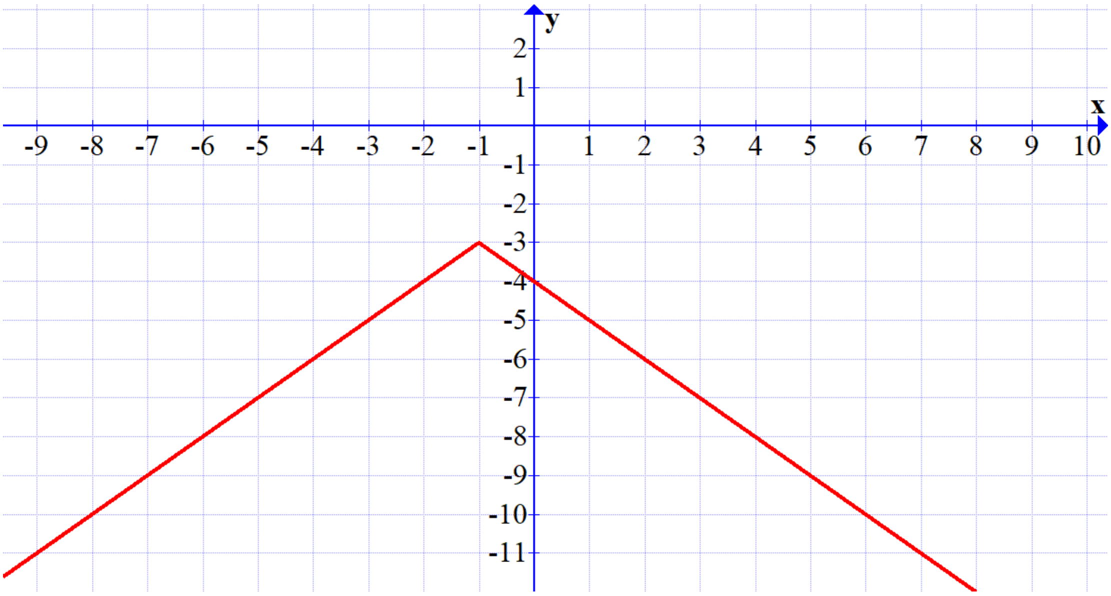
Although this was not part of the question, it is left as an exercise for you to describe the effect of doing $|3f(2|x|-1)|-3$ or of doing $|3f(2|x|-1)-3|$ and to plot the resulting functions.

1.4 How to identify a function from a graph

Here we will go through how to read a graph so that we can write down the equation of the function that it represents.

1.4.1 Example 1: A linear function

Consider the curve on the graph below, whose general form is given by $f(x) = a|x + b| + c$. What are the values of a, b, and c for this curve?



The aspect which we should always focus on first is to identify the original untransformed function. In this case it is $f(x) = |x|$.

Now, in terms of finding a, b, and c, the easiest aspect to focus on first is c, the vertical tranlsation of the function. In our case $|x|$ has been shifted down by 3 units, so $c = -3$. So we have $f(x) = a|x + b| - 3$

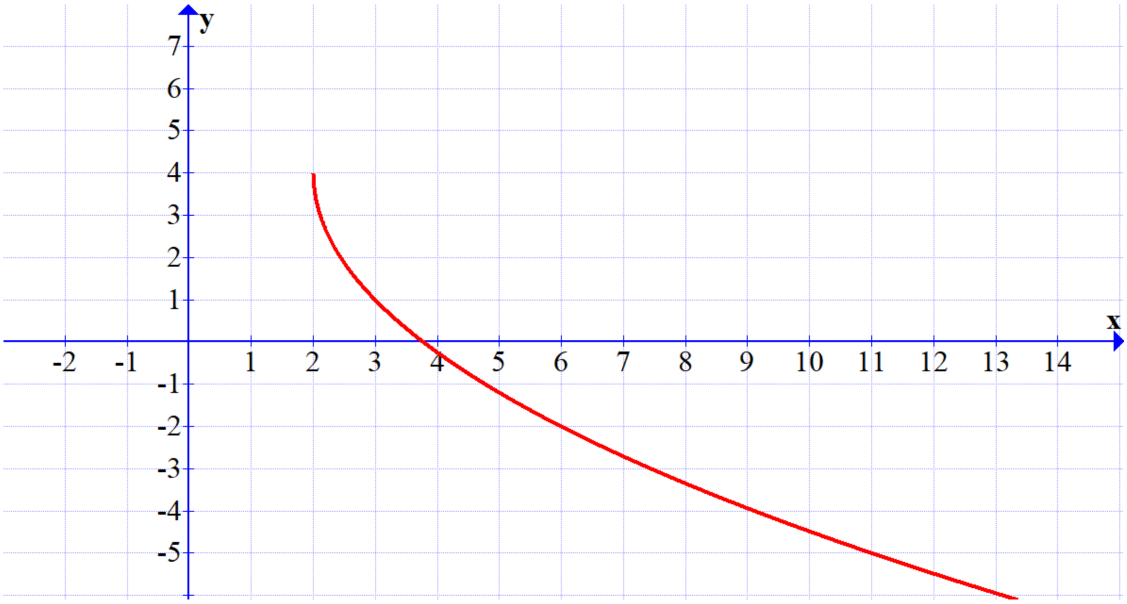
The next easy aspect we can focus on is b, the horizontal translation of the function. In this case $|x|$ has been shifted to the left by 1 unit, so $b = 1$, and we have $f(x) = a|x + 1| - 3$.

Now we find a. Firstly we see that a has to be negative since $|x|$ would be a V looking shape if a were positive. Secondly, we can choose any x and $f(x)$ values and solve $f(x) = a|x + 1| - 3$ to find for a. So when $x = 1, f(x) = -5$ hence $-5 = a|1 + 1| - 3$ gives us $a = -1$.

Hence our function is $f(x) = -|x + 1| - 3$.

1.4.2 Example 2: A square root function

Consider the curve on the graph below, whose general form is given by $f(x) = a\sqrt{x+b} + c$. What are the values of a, b, and c for this curve?



The original untransformed function relating to this curve $f(x) = \sqrt{x}$. We know that this function is an upward curve not a downward curve so we immediately see that a has to be negative. We will come back to finding the value of a later, but I raise this matter here to highlight things that we should learn to recognise most easily and directly.

As before, the easiest aspect to focus on first is c, the vertical translation coefficient. Since $f(x) = \sqrt{x}$ starts at height of $f(x) = 0$ we see here that $c = 4$, so we have $f(x) = a\sqrt{x+b} + 4$.

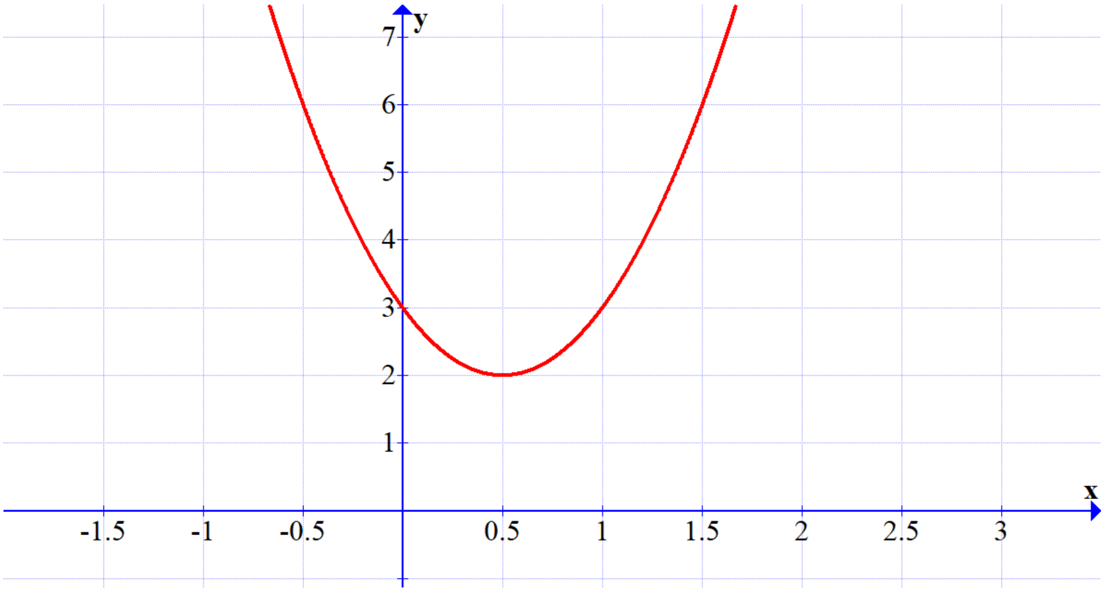
The next aspect we can deal with is coefficient b which hrepresents the horizontal translation of $f(x)$. Normally $\sqrt{x}$ starts at $x = 0$, but in our case $f(x)$ start at $x = 2$, so $b = -2$, and we have $f(x) = a\sqrt{x-2} + 4$.

We can now find coefficient a by choosing a coordinate and then solving for k in $f(x)$. So choosing (3, 1) we have $1 = a\sqrt{3-2} + 4$ which leads to $a = -3$, confirming that a is negative.

Hence our functions is $f(x) = -3\sqrt{x-2} + 4$.

1.4.3 Example 3: A quadratic function

Consider the curve on the graph below, whose general form is given by $f(x) = (ax + b)^2 + c$. What are the values of a, b, and c for this curve?



Be aware that the x-axis scale is not the same as the y-axis scale. There is no need to be confused by this. We will simply rely on the fact that we have an accurate x-axis scale and separately an accurate y-axis scale.

The easiest aspect to focus on first is d, the vertical translation of the curve. From the graph we see that, compared to $f(x) = x^2$, the curve has been shifted upwards by 2 units. So c = 2, and we have $f(x) = (ax + b)^2 + 2$;

The next easiest aspect to focus on might be thought to be b, the horizontal translation of the curve. From the graph we see that, compared to $f(x) = x^2$, the curve has been shifted towards the right by ½ unit. We now have two possible versions for our quadratic. Either $f_1(x) = \left(ax - \frac{1}{2}\right)^2 + 2$ or $f_2(x) = (2ax - 1)^2 + 2$, since both of these will produce a horizontal shift to the right of ½. What we now have to do is choose a coordinate (any coordinate) and solve for a in both equations. For example

For $\left(x, f(x)\right) = (1,3)$

$f_1(x) = \left(ax - \frac{1}{2}\right)^2 + 2$ leads to $a = -\frac{1}{2}, \frac{3}{2}$,	$f_2(x) = (2ax - 1)^2 + 2$ leads to a = 0, 1
so	so
a. $f_1(x) = \left(\frac{3}{2}x - \frac{1}{2}\right)^2 + 2$	c. $f_2(x) = (-1)^2 + 2$
Or	or
b. $f_1(x) = \left(-\frac{1}{2}x - \frac{1}{2}\right)^2 + 2$	d. $f_2(x) = (2x - 1)^2 + 2$

Each answers would have to be tested with another coordinate (such as (1.5, 6)) to see which one was valid.

Another way of approaching this is to set up two simultaneous equation to solve for a and b directly in $f(x) = (ax + b)^2 + 2$. This is a lot quicker. Hence, choosing two coordinates (0, 3) and (1, 3) we end up with

$$3 = (a \times 0 + b)^2 + 2$$

and

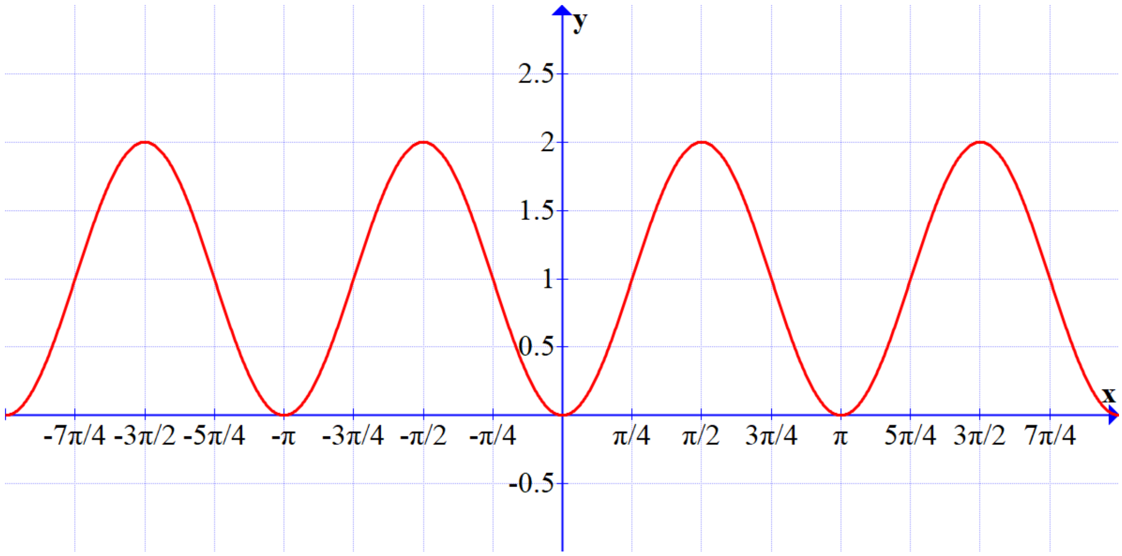
$$2 = \left(a \times \frac{1}{2} + b\right)^2 + 2.$$

Solving this gives $a = 2, b = -1$ and $a = -2, b = 1$. Either of these answers are correct since they produce equivalent quadratics (this can be seen by testing any other coordinate such as (1.5, 6))

Hence our quadratic can be written as $f(x) = (2x - 1)^2 + 2$.

1.4.4 Example 4: A sine function

Consider the curve on the graph below, whose form is given by $f(x) = \sin(ax + b) + c$. What are the values of a, b, c, and c for this curve?



The easiest aspect to focus on first is d, the vertical translation of the curve. From the graph we see that, compared to $f(x) = \sin x$, the curve has been shifted upwards by 1 unit. So $c = 1$, and we have $f(x) = \sin(ax + b) + 1$.

It might be thought the easiest thing to do next woud be to find b, the horizontal translation of the function. In the above graph the sine curve has been shifted right by $\pi/4$, so it might seem as if $b = -\pi/4$. To see why this is the kind of distance we are looking for note that when finding the horizontal translation of a trig function we always want to know how far left or right the curve has moved compared to the where the trig function would be (0, 0). We know that the point on the sine curve in the graph above which would be normally be located at (0, 0) is now located at the point [$\pi/4$, 1], hence we might believe that $b = -\pi/4$.

However, this is not the correct value of b because we have not taken into account the value a. This latter value has a direct effect the amount of horizontal translation (this is true for the transformation of any function, not just sine curves).

In order to find the correct value of b we could repeat the analysis of the previous example and choose two coordinates, say (0, 0) and ($\pi/2$, 2), then solve $f(x) = \sin(ax + b) + 1$ to find a and b. Instead we will go through analysing the graph from a different perspective.

As such, let us firstly find out the value a. This can be found directly from the graph by seeing that the horizontal distance between two adjacent peaks is π. Since the distance between adjacent peaks for a standard sine curve is 2π this means that we can fit two $f(x)$ curves within the standard sine curve. Hence our value is a = 2 and we now have $f(x) = \sin(2x + b) + 1$.

Now we come to finding b given that a = 2. The summary of why the value of a affects the value of b is that, as the sine wave gets compressed by a factor of 2 when transforming from x to 2x, so the horizontal translation effect gets compressed by the same "1 over" the factor, i.e. from b to $b/2$. Hence $b/2$ = –π/4, implying that b = –π/2.

To see why this is the case notice that the horizontal translation seen in the graph above is done with respect to "2x" not "x". In other words we are transforming from $f(x) = \sin(2x) + 1$ to $f(x) = \sin\big((2x) + b\big) + 1$, not from $f(x) = \sin(x) + 1$ to $f(x) = \sin(x + b) + 1$. So, in the graph above, the x-axis actually represents a "2x" axis, and it is on this "2x" axis that we see the horizontal translation to be –π/4.

But we want the value of horizontal translation with respect to the x-axis not the "2x" axis. In rewriting $f(x) = \sin(2x + b) + 1$ as $f(x) = \sin 2(x + b/2) + 1$ we can see that the horizontal shift on x is now $b/2$. Hence, it is $b/2$ that equals –π/4, not b that equals –π/4. Therefore b = –π/2.

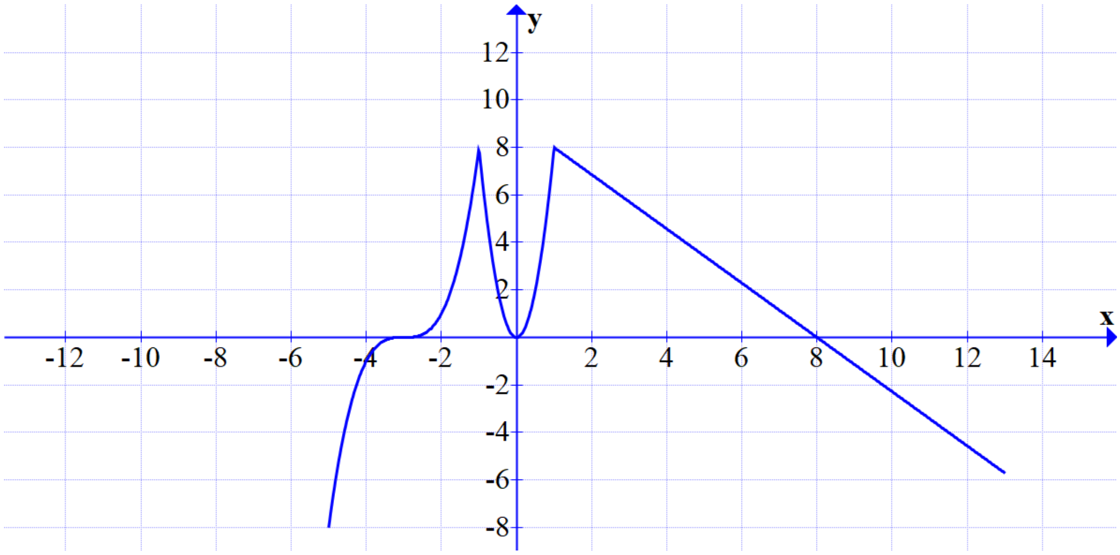
As mentioned previously, this effect applies to any function $f(x)$ that is transformed to $f(ax + b)$. In general, for any horizontal translation of k units seen visually on a graph we have b/a = k, implying that b = a×k.

So in summary we note that the value a horizontal translation (here, b) is affected by the value of the compression/stretching factor (here, a), so that we have to find this latter value first before finding the former value.

Hence our sine function can be written as $f(x) = \sin(2x - \pi/2) + 1$.

1.4.5 Example 5: A piecewise function

Consider a function $f(x)$ in the interval $[-5,13]$ whose curve is as in the graph below

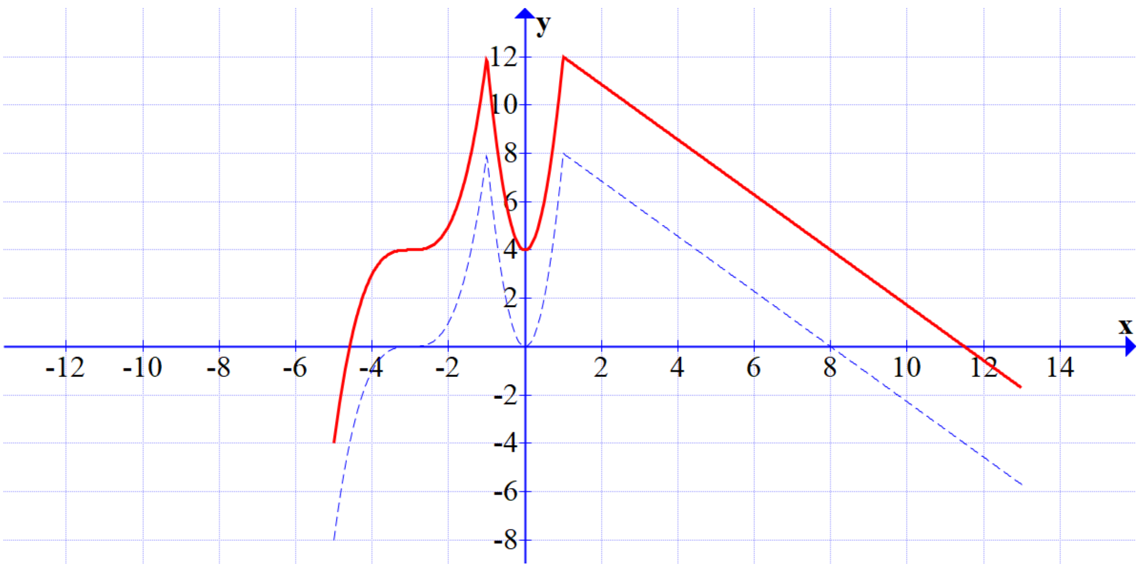


How is this curve transformed under the following transformation?

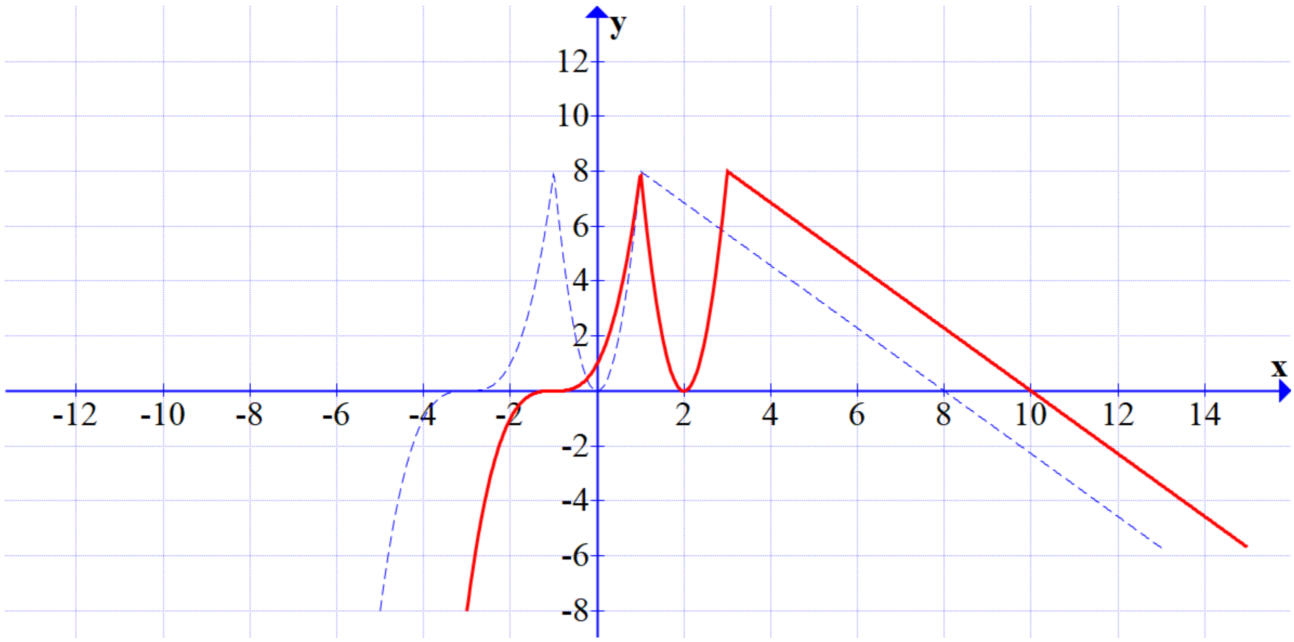
1. $f(x) + 4$,
2. $f(x - 2)$,
3. $f(3x)$,
4. $f(2x - 4) + 2$,
5. $-f(x - 1) + 2$,
6. $f(-x + 1)$,
7. $|f(x)|$
8. $f(|x|)$

<u>*Solutions*</u>

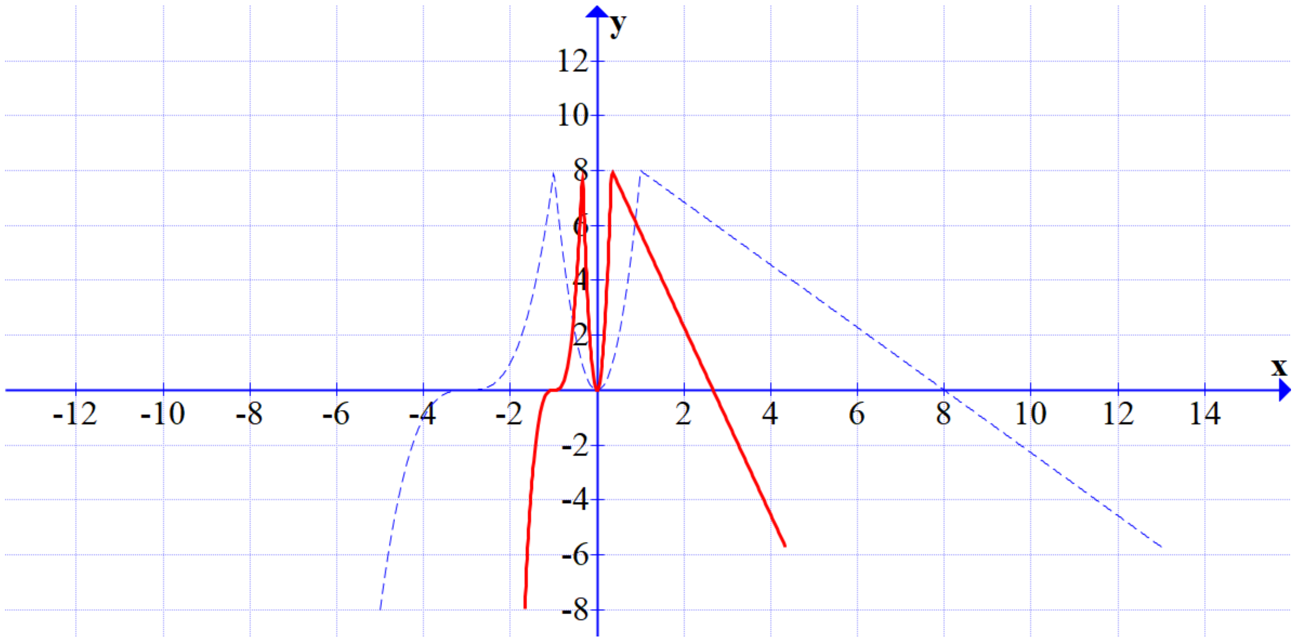
The effect of $f(x) + 4$ will be to simply translate the function upwards by 2 units, as shwon in the graph below. In all graphs to be shown from this point onwards the blue-dashed curve will represent the orignial untransformed function $f(x)$ and the thick red line will rpresent the transformed curve. Hence $f(x) + 2$ looks like this:



The effect of $f(x-2)$ will be to translate the function to the left by 2 units:



For $f(3x)$ the "3" acts like the frequency of the curve. This means that we should be able to fit three widths of $f(3x)$ within the original untransformed width of $f(x)$. For this to be able to happen the width of $f(x)$, i.e. the range from −5 to 13, will have to be shortened by one-third. This means that every value of $f(x)$ will be divided by 3. Also since $f(3x) = 0$ when x = 0 the curve will remain fixed at x = 0 (this is true also because there is no horizontal translation being done on $f(x)$). This is shown in the graph below:

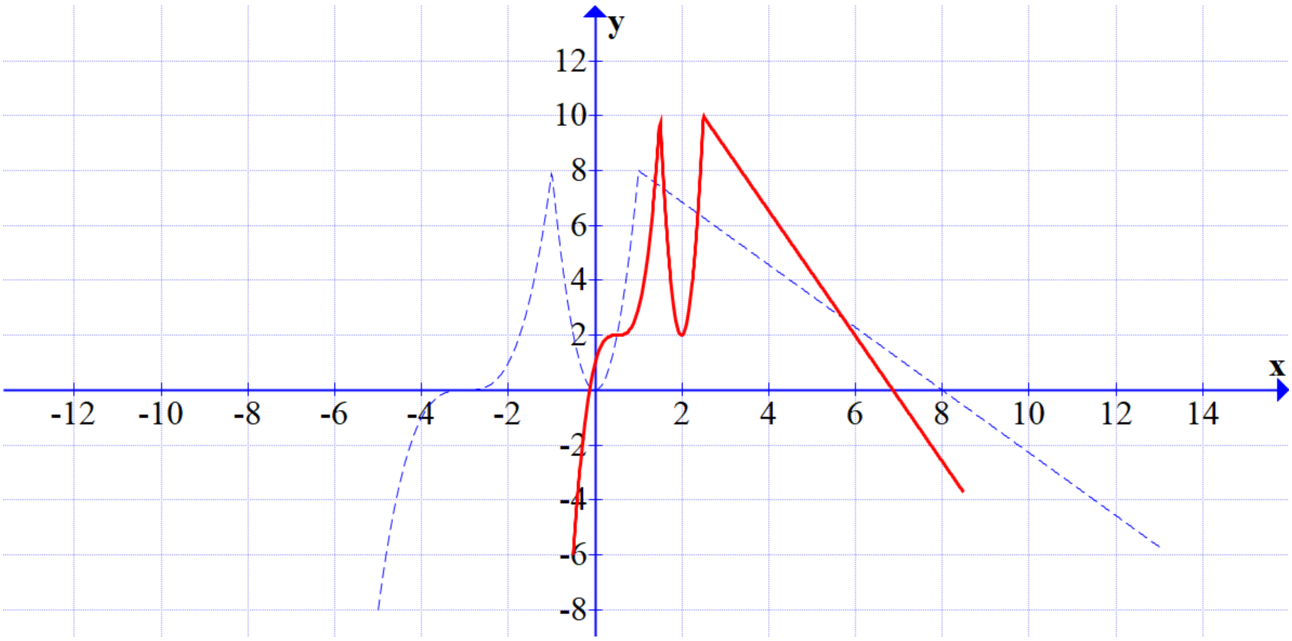


Although you might not be able to see it on the graph, the width of $f(3x)$ is 13/3 – (−5/3) = 18/3 = 6 which is one-third of rthe interval [−5, 13].

The effect of the transformation $f(2x-4)+2$ is just a combination of the previous three examples. There is a vertical and horizontal translation along with a frequency effect which causes a horizontal compression. On this occasion we have to be careful about how we read the amount of horizontal translation.

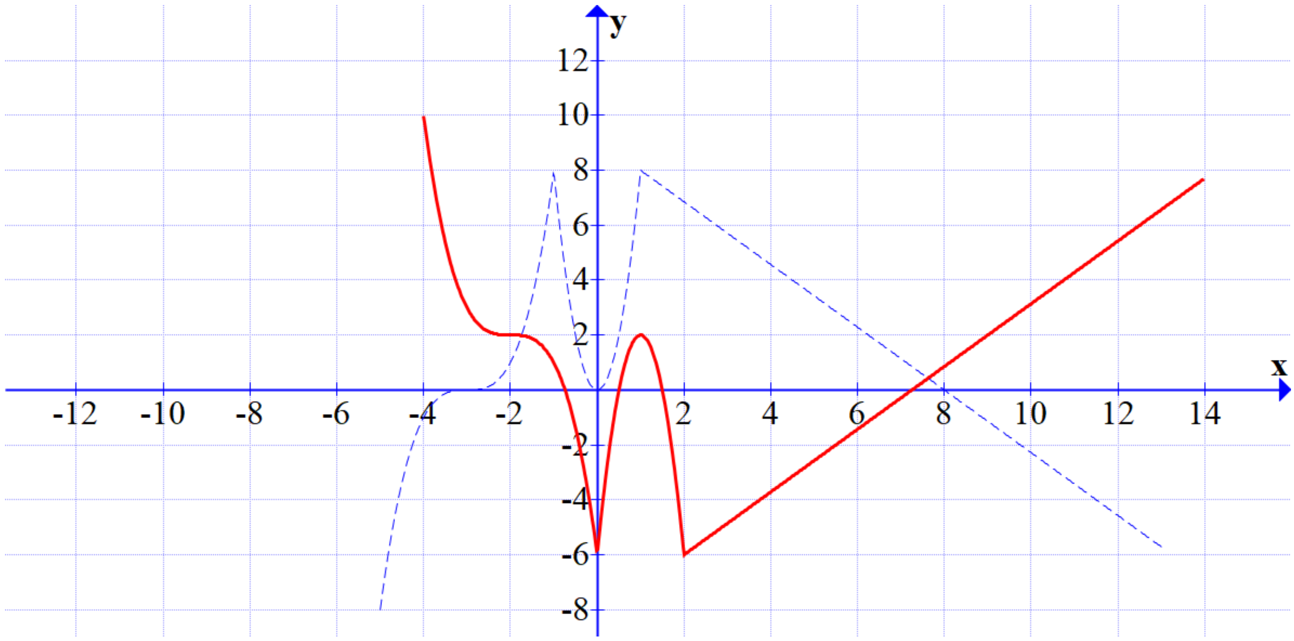
To explain this consider a function such as $g(x-4)$. Such a function means that the original function $g(x)$ is simply shifted rightward by 4 units. But in the function $g(2x-4)$ the function isn't shifted by 4 units because there is the effect of the frequency of 2 units which affects this shift. So if $g(2x)$ means that we can fit two $g(2x)$ functions in the interval of $g(x)$ then the interval of $g(2x)$ has been halved compared to that of $g(x)$. This means that the rightward shift of $g(2x-4)$ has also been halved compared to that of $g(x-4)$.

Hence $f(2x-4)$ has a frequency of 2 units and a rightward shift of 2 units as shown below:



For the function $-f(x-1)+2$ we have the usual transformations of vertical and horizontal translations and now, because of the minus sign, we have a reflection in the x-axis. The best way to determine how $f(x)$ changes under this transformation is to start by looking at $-f(x)$ and then perform the relevant translations.

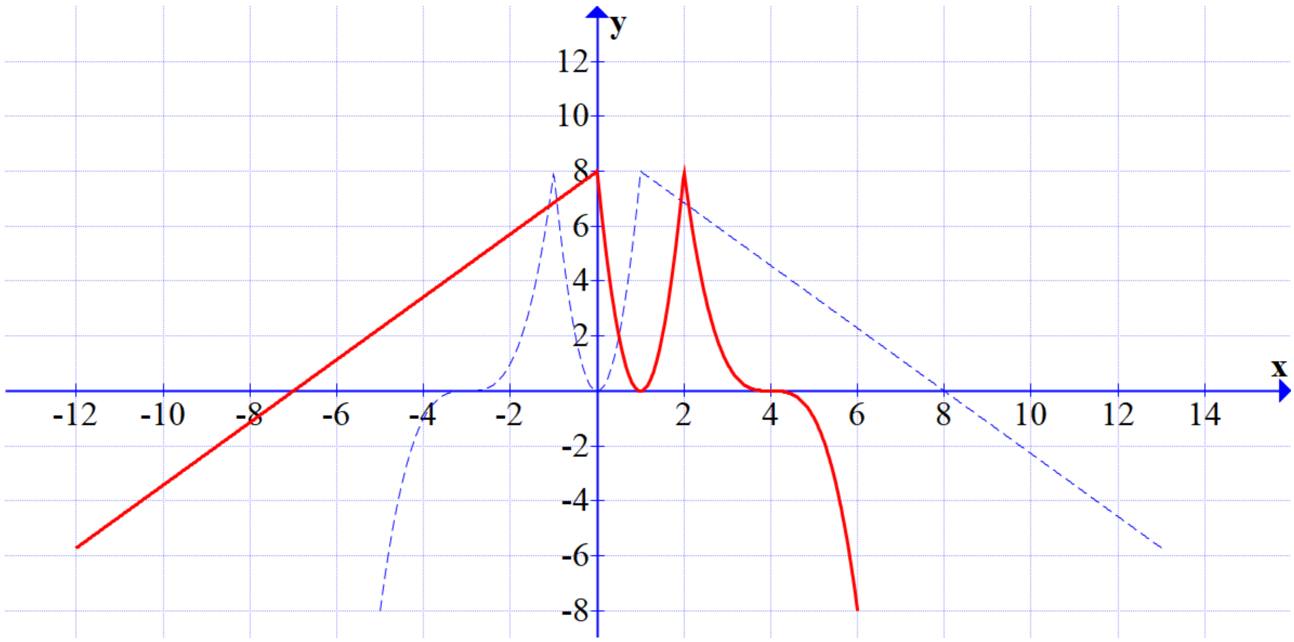
So, reflecting $f(x)$ in the x-axis, and then shifting right by 1 unit, and then shifting up by 2 units will give a graph which looks like the one shown below:



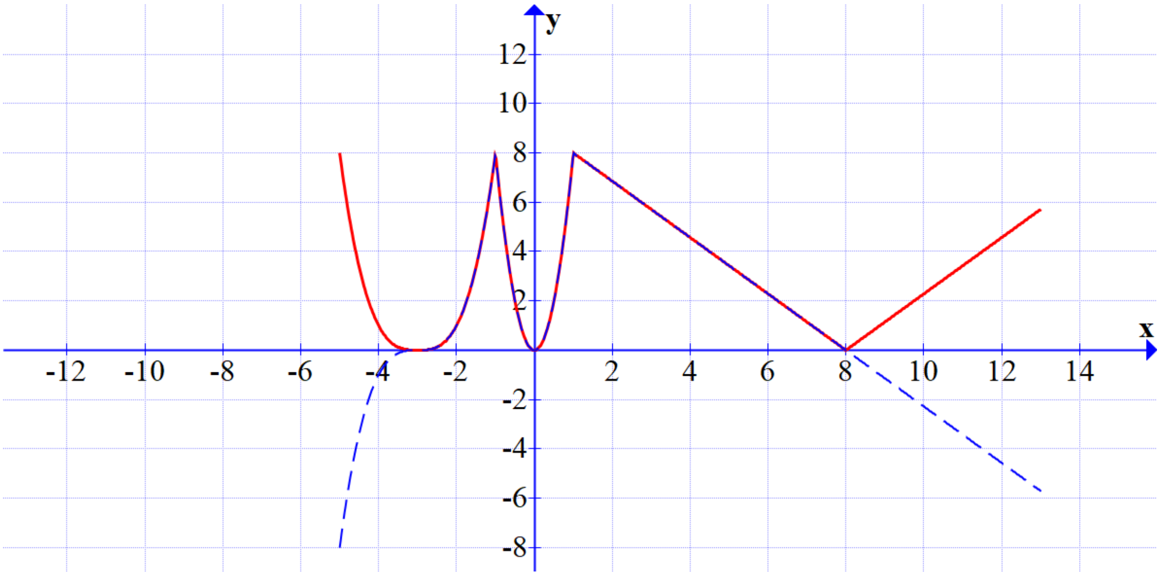
For the function $f(-x+1)$ we have the usual transformation of a horizontal translation and, because of the minus sign, we now have a reflection in the y-axis. As before it is best to first consider the transformation $f(-x)$ and then perform the horizontal translation.

So, $f(-x)$ will reflect $f(x)$ in the y-axis. Since the curve has now been relfected in the y-axis, what would be a leftward shift of 1 unit given by the "+1" now becomes a right shift of 1 unit (i.e. everything is now "backwards" or reversed because of the reflection in the y-axis).

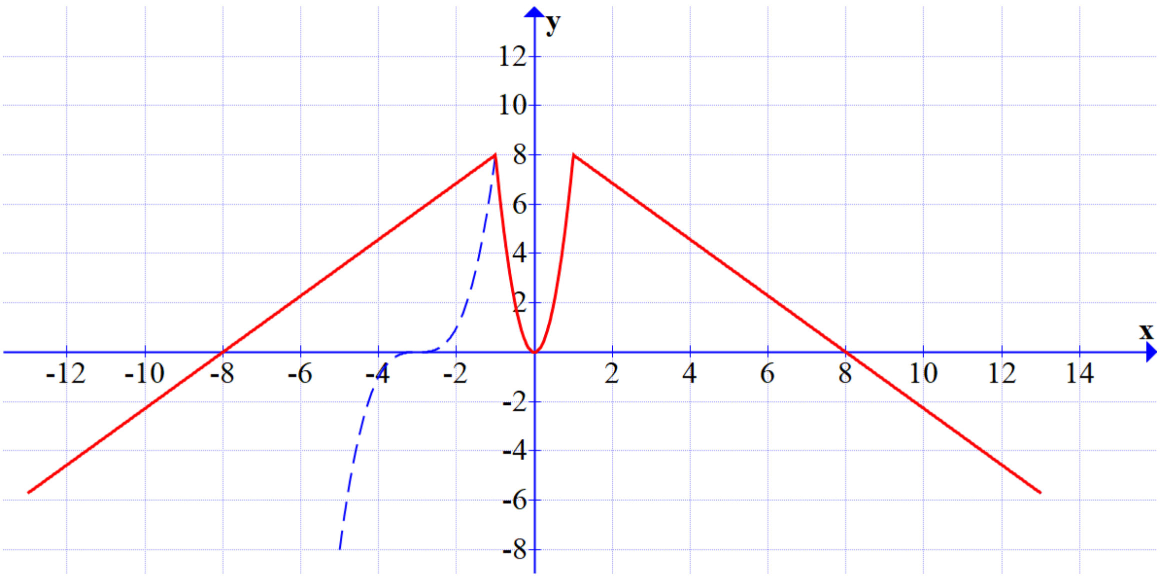
Hence the graph of $f(-x+1)$ is



For the function $|f(x)|$ all values of $f(x)$ which were negative now become positive. This means that we end up with a reflection in the x-axis of only the negatie values of $f(x)$. This is seen below:



For the function $f(|x|)$ all values of x which were negative now become positive. This means that we end up evaluating things like $f(-3), f(-2), f(-1)$ as if they were $f(3), f(2), f(1)$. This implies that the shape of the curve on the negative side of the x-axis will look like a backward version (i.e. a reflection) of the curve on the positive side of x-axis. This is seen in the graph below:



Notice here that because of the nature of the transformation $f(|x|)$ we have completely lost everything that is on the left hand side of the y-axis, for it to be replaced by what is on the right hand side of the y-axis (Can you see why we still seem to have the left hand part of the x^2 shaped curve? Hint: It is not the left hand part of the original x^2 shaped curve).

1.5 Summary

	$f(x) + k$	$f(x + k)$
$f(x)$ values	Translate upwards if $k > 0$; Translate downwards if $k < 0$	Translate leftwards if $k > 0$; Translate rightwards if $k < 0$

	$k.f(x)$	$f(kx)$
$f(x)$ values	$k.f(x)$ stretches $f(x)$ away from x-axis as k increase from 0 $k.f(x)$ compresses $f(x)$ towards x-axis as k decreases to 0	$k.f(x)$ stretches $f(x)$ away from y-axis as k decrease to 0 $k.f(x)$ compresses $f(x)$ towards y-axis as k increase from 0

	$\|f(x)\|$	$f(\|x\|)$
$f(x)$ values	Reflects in the x-axis all negative $f(x)$ values	Reflects in the y-axis all values of $f(x)$ which are on the positive side of the x-axis